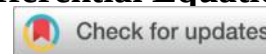




On the Solvability of a Nonlinear Hadamard Fractional Differential Equation with an Integral Boundary Condition



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Abstract

The aim of this work is to investigate the existence, uniqueness, and positivity of a solution to a nonlinear Hadamard fractional differential equation supplemented with an integral boundary condition. Our approach leverages several key theorems from nonlinear functional analysis: the Leray-Schauder nonlinear alternative, the Banach contraction mapping principle, and the Guo-Krasnosel'skii fixed point theorem on cone expansion and compression. Finally, we provide illustrative examples to demonstrate the applicability of our theoretical findings.

Keywords: Cone, fixed point theorem, Hadamard fractional differential equations, Integral condition.

Mathematics Subject Classifications: 34B10,34B15,26A33.

1. Introduction

The ability of fractional calculus to capture non-local effects, such as memory and hereditary properties, makes it exceptionally well-suited for describing complex phenomena in materials science, viscoelasticity, and anomalous transport. This has driven intense interest in the analysis of fractional-order boundary value problems (BVPs). A central challenge in this field is proving the existence and uniqueness of solutions, especially for nonlinear equations where analytical solutions are often unattainable. To address this, researchers frequently employ methods from nonlinear functional analysis, with fixed point theorems (e.g., Banach, Schaefer, Krasnoselskii) serving as a primary technique for proving existence results.

A significant portion of this work, detailed in [1,4,6,10,11,12,14,16,20] and their citations, focuses specifically on ensuring the existence of positive solutions, which are physically meaningful in many applications.

While the literature on this class of problems is predominantly based on the Riemann-Liouville and Caputo definitions of fractional calculus, other formulations offer unique advantages. A notable example is the Hadamard fractional derivative, which is characterized by the inclusion of a logarithmic function of arbitrary exponent in its kernel. This structure makes the Hadamard derivative a natural choice for modeling phenomena exhibiting semi-infinite or logarithmic scaling, as it is invariant under dilation scaling. Key references for the theory and application of Hadamard derivatives include [2,3,7,9,13,17,18,19] .

In [12], authors studied the existence of at least three positive solutions to the following singular boundary value problem:

$$\begin{cases} {}^C D_{0^+}^\alpha x(\varsigma) + f(\varsigma, x(\varsigma), x'(\varsigma)) = 0, & 1 < \varsigma < 1. \\ x(\varsigma) = x''(\varsigma) = 0, & x'(1) = \sum_{j=1}^{\infty} \eta_j x(\xi_j), \end{cases}$$

where $2 < \alpha \leq 3$, $\eta_j \leq 0$, $0 < \xi_1 < \xi_2 < \dots < \xi_{j-1} < \xi_j < \dots < 1$, ($j=1,2,\dots$) and ${}^C D_{0^+}^\alpha$ is the standard Caputo derivative.

Wengui Yang [18] applied the Leray-Schauder nonlinear alternative and Krasnosel'skii's fixed point theorem to prove the existence of positive solutions for a class of coupled semipositone Hadamard fractional differential equations with integral boundary conditions.

In [2], authors investigated the existence criteria for the following problem:

$$\begin{cases} ({}^H D^\alpha + \lambda {}^H D^{\alpha-1})x(\varsigma) = f(\varsigma, x(\varsigma)), & 1 < \alpha \leq 2, 1 < \varsigma < T. \\ x(1) = 0, & x(T) = \sum_{j=1}^m \beta_j x(\varsigma_j), \end{cases}$$

where ${}^H D^\alpha$ denotes the Hadamard fractional derivative of order α , $f : [1, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $\lambda \in \mathbb{R}^+$, ς_j , $j=1,2,\dots,m$, are given points with $1 \leq \varsigma_1 \leq \dots \leq \varsigma_m < T$, and β_j are appropriate real numbers.

Let E be the Banach space of continuous functions $C[1, e]$, endowed with the norm $\|\varpi\| = \max_{\varsigma \in [1, e]} |\varpi(\varsigma)|$.

Motivated by the work discussed above and others we investigate to the following nonlinear Hadamard fractional differential equation with integral boundary condition:

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$$\begin{cases} {}^H D^q \varpi(\varsigma) + f(\varsigma, \varpi(\varsigma)) = 0, & \varsigma \in (a, e). \\ \varpi(a) = \varpi'(a) = 0, & x(1) = \eta \int_1^e \varpi(\varsigma) dt, \end{cases}$$

where ${}^H D^q$ is the Hadamard fractional derivative of fractional order q , $f \in C([1, e] \times \mathbb{R}_+, \mathbb{R}_+)$, $n \geq 3$, $q \in n-1, n$ is a real number, and $0 < \eta < 2$.

The organization of this paper is as follows. We begin in Section 2 with preliminary material, including key definitions, lemmas, and a study of the Green's function properties. The section also outlines the fixed point theorems employed in subsequent sections. Sections 3 and 4 are devoted to stating and proving the main results on the existence, uniqueness, and positivity of solutions, achieved via the Leray-Schauder nonlinear alternative, the Banach contraction principle, and the Guo-Krasnosel'skii fixed point theorem. Concluding examples that illustrate the applicability of our theorems are given in Section 5.

2. Preliminaries

We introduce some necessary definitions, lemmas and theorems which will be used in this paper.

Definition 1. [8] *The fractional integral*

$$I_{0^+}^\alpha f(\varsigma) = \frac{1}{\Gamma(\alpha)} \int_0^\varsigma \frac{f(s)}{(\varsigma - s)^{1-\alpha}} ds,$$

where $\alpha > 0$, is called Riemann-Liouville fractional integral of order α of a function $f : (0, +\infty) \rightarrow \mathbb{R}$ and $\Gamma(\cdot)$ is the gamma function.

Definition 2. [8] *The Riemann-Liouville fractional derivative of order $\alpha > 0$, of a continuous function $f : (0, +\infty) \rightarrow \mathbb{R}$ is given by*

$$D_{0^+}^\alpha f(\varsigma) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^\varsigma (\varsigma - s)^{n-\alpha-1} f(s) ds.$$

Where $\Gamma(\cdot)$ is the gamma function, and $n = [\alpha] + 1$, with $[\alpha]$ denoting the greatest integer less than or equal to α . It is assumed that the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 3. [8] *The Hadamard fractional integral of order $\alpha > 0$, for a continuous function $f : [1, +\infty) \rightarrow \mathbb{R}$ is given by*

$${}^H \mathfrak{I}^\alpha \mathbf{F}(\varsigma) = \frac{1}{\Gamma(\alpha)} \int_1^\varsigma \left(\log \frac{\varsigma}{s} \right)^{\alpha-1} \mathbf{F}(s) \frac{ds}{s}.$$

Definition 4. [8] *Let $\alpha > 0$ and $[\alpha]$ its integer part. The Hadamard fractional derivative of order α of the function $f : [1, +\infty) \rightarrow \mathbb{R}$ is defined as*

$${}^H D^\alpha \mathbf{F}(\varsigma) = \left(\varsigma \frac{d}{dt} \right)^n \frac{1}{\Gamma(n-\alpha)} \int_1^\varsigma \left(\log \frac{\varsigma}{s} \right)^{n-\alpha-1} \mathbf{F}(s) \frac{ds}{s}.$$

Where $n = [\alpha] + 1$.

Lemma 5. [8] Assume that $\varpi \in C(0,1) \cap L^1(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then

$$I_{0^+}^\alpha D_{0^+}^\alpha \varpi(\varsigma) = \varpi(\varsigma) + c_1 \varsigma^{\alpha-1} + c_2 \varsigma^{\alpha-2} + \dots + c_n \varsigma^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$; $n = [\alpha] + 1$.

Definition 6. [15] Let $(E, \|\cdot\|)$ be a Banach space and $\mathbf{A} : E \rightarrow E$. The operator $\mathbf{A}$ is a contraction operator if there is an $\alpha \in (0,1)$ such that $\varpi, v \in E$ imply

$$\|\mathbf{A}v\| \leq \alpha \|\varpi - v\|.$$

Theorem 7. [15] (Banach) Let K be a nonempty closed convex subset of a Banach space E and $\mathbf{A} : K \rightarrow K$ be a contraction operator. Then there is a unique $x \in K$ with $\mathbf{A}x = x$.

Theorem 8. [5] (Guo - Krasnosel'skii) Let E be a Banach space, and let $K \subset E$, be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let

$$\mathbf{A} : K \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow K,$$

be a completely continuous operator. In addition suppose either

$$(i) \|\mathbf{A}\varpi\| \leq \|\varpi\|, \varpi \in K \cap \partial\Omega_1, \text{ and } \|\mathbf{A}\varpi\| \geq \|\varpi\|, \varpi \in K \cap \partial\Omega_2; \text{ or}$$

$$(ii) \|\mathbf{A}\varpi\| \geq \|\varpi\|, \varpi \in K \cap \partial\Omega_1, \text{ and } \|\mathbf{A}\varpi\| \leq \|\varpi\|, \varpi \in K \cap \partial\Omega_2,$$

holds. Then $\mathbf{A}$ has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Lemma 9. let $y \in C[1, e]$. Then the Hadamard fractional boundary value problem

$$\begin{cases} {}^H D^q \varpi(\varsigma) + y(\varsigma) = 0, & \varsigma \in (a, e). \\ \varpi(1) = \varpi'(1) = 0, & \varpi(e) = \eta \int_1^e \varpi(\varsigma) dt, \end{cases}$$

has a unique solution, given by

$$\varpi(\varsigma) = \int_1^e H(\varsigma, s) y(s) \frac{ds}{s}.$$

Where,

$$H(\varsigma, s) = G(\varsigma, s) + \frac{\eta (\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(\varsigma, s) dt,$$

$$G(\varsigma, s) = \frac{1}{\Gamma(q)} \begin{cases} (\log \varsigma)^{q-1} (1 - \log s)^{q-1} - \left(\log \frac{\varsigma}{s}\right)^{q-1}, & 1 \leq s \leq \varsigma \leq e, \\ (\log \varsigma)^{q-1} (1 - \log s)^{q-1}, & 1 \leq \varsigma \leq s \leq e. \end{cases}$$

and, $\Lambda = 1 - \eta \int_1^e (\log \varsigma)^{q-1} dt > 0$

Proof . The solution of the Hadamard differential equation in (2.1) can be written as the equivalent integral equation

$$\varpi(\varsigma) = C_1 (\log \varsigma)^{q-1} + C_2 (\log \varsigma)^{q-2} + C_3 (\log \varsigma)^{q-3} - \frac{1}{\Gamma(q)} \int_1^\varsigma \left(\log \frac{\varsigma}{s}\right)^{q-1} y(s) \frac{ds}{s}.$$

From the boundary condition $\varpi(1) = \varpi'(1) = 0$ we get $C_3 = C_2 = 0$. And,

$$\varpi(\varsigma) = C_1 (\log \varsigma)^{q-1} - \frac{1}{\Gamma(q)} \int_1^\varsigma \left(\log \frac{\varsigma}{s}\right)^{q-1} y(s) \frac{ds}{s}.$$

From the boundary condition $\varpi(e) = \eta \int_1^e \varpi(\varsigma) dt$, we deduce that

$$\varpi(e) = C_1 - \frac{1}{\Gamma(q)} \int_1^e \left(\log \frac{e}{s}\right)^{q-1} y(s) \frac{ds}{s} = \eta \int_1^e \varpi(\varsigma) dt$$

$$C_1 = \eta \int_1^e \varpi(\varsigma) dt + \frac{1}{\Gamma(q)} \int_1^\varsigma \left(\log \frac{e}{s}\right)^{q-1} y(s) \frac{ds}{s}$$

Then

$$\begin{aligned} \varpi(\varsigma) &= \eta (\log \varsigma)^{q-1} \int_1^e \varpi(\varsigma) dt + \frac{(\log \varsigma)^{q-1}}{\Gamma(q)} \int_1^e \left(\log \frac{e}{s}\right)^{q-1} y(s) \frac{ds}{s} \\ &\quad - \frac{1}{\Gamma(q)} \int_1^\varsigma \left(\log \frac{\varsigma}{s}\right)^{q-1} y(s) \frac{ds}{s}. \end{aligned}$$

So,

$$\begin{aligned} \varpi(\varsigma) &= \eta (\log \varsigma)^{q-1} \int_1^e \varpi(\varsigma) dt + \frac{(\log \varsigma)^{q-1}}{\Gamma(q)} \int_1^\varsigma \left(\log \frac{e}{s}\right)^{q-1} y(s) \frac{ds}{s} \\ &\quad + \frac{(\log \varsigma)^{q-1}}{\Gamma(q)} \int_\varsigma^e \left(\log \frac{e}{s}\right)^{q-1} y(s) \frac{ds}{s} - \frac{1}{\Gamma(q)} \int_1^\varsigma \left(\log \frac{\varsigma}{s}\right)^{q-1} y(s) \frac{ds}{s}. \end{aligned}$$

$$\begin{aligned} \varpi(\varsigma) &= \eta (\log \varsigma)^{q-1} \int_1^e \varpi(\varsigma) dt + \frac{1}{\Gamma(q)} \int_\varsigma^e (\log \varsigma)^{q-1} (1 - \log s)^{q-1} y(s) \frac{ds}{s} \\ &\quad + \frac{1}{\Gamma(q)} \int_1^\varsigma \left((\log \varsigma)^{q-1} (1 - \log s)^{q-1} - \left(\log \frac{\varsigma}{s}\right)^{q-1} \right) y(s) \frac{ds}{s}. \end{aligned}$$

$$\varpi(\varsigma) = \eta (\log \varsigma)^{q-1} \int_1^e \varpi(\varsigma) dt + \int_a^e G(\varsigma, s) y(s) \frac{ds}{s}.$$

Integrating this result with respect to ς from 1 to e , we obtain

$$\int_1^e \varpi(\varsigma) dt = \int_1^e \eta(\log \varsigma)^{q-1} dt \int_1^e \varpi(\varsigma) dt + \int_1^e \int_1^e G(\varsigma, s) y(s) \frac{ds}{s} dt.$$

$$\int_1^e \varpi(\varsigma) dt = \frac{1}{1 - \int_a^e \eta(\log \varsigma)^{q-1} dt} \int_1^e \int_1^e G(\varsigma, s) y(s) \frac{ds}{s} dt.$$

Therefore,

$$\begin{aligned} \varpi(\varsigma) &= \frac{\mu(\log \varsigma)^{q-1}}{1 - \int_a^e \mu(\log \varsigma)^{q-1} dt} \int_1^e \int_1^e G(\varsigma, s) y(s) \frac{ds}{s} dt + \int_1^e G(\varsigma, s) y(s) \frac{ds}{s}, \\ &= \int_1^e \left(G(\varsigma, s) + \frac{\mu(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(\varsigma, s) dt \right) y(s) \frac{ds}{s}, \\ &= \int_1^e H(\varsigma, s) y(s) \frac{ds}{s}. \end{aligned}$$

Where $G(\varsigma, s)$ is defined by (2.3). The proof is complete.

Now we give some properties of the Green function.

Lemma 10. [19] *The function $G(\varsigma, s)$ defined by (2.3) satisfies the following properties*

(i) $G(\varsigma, s) \geq 0$ and $G(\varsigma, s) \in C([1, e] \times [1, e], \mathbb{R}_+)$.

(ii)

$$k(\varsigma) \hat{k}(\varsigma) \leq \Gamma(q) G(\varsigma, s) \leq (q-1) \hat{k}(\varsigma), \text{ for } \varsigma, s \in [1, e]$$

(iii)

$$k(\varsigma) \hat{k}(\varsigma) \leq \Gamma(q) G(\varsigma, s) \leq (q-1) k(\varsigma), \text{ for } \varsigma, s \in [1, e]$$

Where, $k(\varsigma) = (\log \varsigma)^{q-1} (1 - \log \varsigma)$ and $\hat{k}(\varsigma) = (\log \varsigma) (1 - \log \varsigma)^{q-1}$, for $q > 2$, $\varsigma, s \in [1, e]$

Combining Lemma 9 and Lemma 10, we obtain

$$\begin{aligned} &\frac{1}{\Gamma(q)} \left(k(\varsigma) + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_a^e k(\varsigma) dt \right) \hat{k}(\varsigma) \\ &\leq H(\varsigma, s) \leq \frac{q-1}{\Gamma(q)} \left(1 + \frac{1}{\Lambda} \int_1^e dt \right) \hat{k}(\varsigma). \end{aligned}$$

Lemma 11. *The function $G(\varsigma, s)$ defined by (2.3) satisfies the following properties*

(i) $G(\varsigma, s) \geq 0$ and $G(\varsigma, s) \in C([1, e] \times [1, e], \mathbb{R}_+)$.

(ii) $\forall 1 \leq s, \varsigma \leq e$

$$(\log \varsigma)^{q-1} G(e, s) \leq G(\varsigma, s) \leq G(e, s).$$

Proof. The continuity of G is easily checked.

If $1 \leq \varsigma \leq s \leq e$, it is easy to see that $G(\varsigma, s) \geq 0$, and

$$(\log \varsigma)^{q-1} G(e, s) = G(\varsigma, s) \leq G(e, s)$$

If $1 \leq s \leq \varsigma \leq e$, we have

$$\begin{aligned} \Gamma(q)G(\varsigma, s) &= (\log \varsigma)^{q-1} (1 - \log s)^{q-1} - \left(\log \frac{\varsigma}{s} \right)^{q-1}, \\ &= (\log \varsigma)^{q-1} (1 - \log s)^{q-1} - \left(1 - \frac{\log s}{\log \varsigma} \right)^{q-1} (\log \varsigma)^{q-1}, \\ &\geq (\log \varsigma)^{q-1} (1 - \log s)^{q-1} - (1 - \log s)^{q-1} (\log \varsigma)^{q-1} = 0. \end{aligned}$$

Then

$$G(\varsigma, s) \geq 0.$$

And,

$$\begin{aligned} &\frac{d}{dt} \left[(\log \varsigma)^{q-1} (1 - \log s)^{q-1} - \left(\log \frac{\varsigma}{s} \right)^{q-1} \right] \\ &= (\alpha - 1) \left[(\log \varsigma)^{q-2} (1 - \log s)^{q-1} - \left(\log \frac{\varsigma}{s} \right)^{q-2} \right], \\ &\geq 0, \end{aligned}$$

which implies that $G(\varsigma, s)$ is the monotone nondecreasing function, so

$$G(\varsigma, s) \leq G(e, s).$$

On the other hand,

$$\begin{aligned} \frac{G(\varsigma, s)}{G(e, s)} &= \frac{(\log \varsigma)^{q-1} (1 - \log s)^{q-1} - \left(\log \frac{\varsigma}{s} \right)^{q-1}}{(1 - \log s)^{q-1} - (1 - \log s)^{q-1}} \\ &\geq \frac{(\log \varsigma)^{q-1} (1 - \log s)^{q-1} - (\log \varsigma - \log \varsigma \log s)^{q-1}}{(1 - \log s)^{q-1} - (1 - \log s)^{q-1}} = (\log \varsigma)^{q-1}, \end{aligned}$$

which implies

$$G(\varsigma, s) \geq (\log \varsigma)^{q-1} G(e, s).$$

Finally

$$(\log \varsigma)^{q-1} G(e, s) \leq G(\varsigma, s) \leq G(e, s), \quad \forall 1 \leq s, \varsigma \leq e.$$

The proof is complete.

We now note that ϖ is the solution of problem (1.1) if and only if ϖ is a fixed point of the operator

$$\begin{aligned}
Tu(\varsigma) &= \int_1^e H(\varsigma, s) f(s, \varpi(s)) \frac{ds}{s} \\
Tu(\varsigma) &= \int_1^e \left(G(\varsigma, s) + \frac{\mu(\log \varsigma)^{q-1}}{\Lambda} \int_a^e G(\varsigma, s) dt \right) f(s, \varpi(s)) \frac{ds}{s}, \\
&= \int_1^e \left(G(\varsigma, s) f(s, \varpi(s)) + \frac{\mu(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(\varsigma, s) f(s, \varpi(s)) dt \right) \frac{ds}{s}
\end{aligned}$$

The operator T is continuous in view of continuity of $G(\varsigma, s)$ and $f(\varsigma, \varpi(\varsigma))$, $\forall 1 \leq s, \varsigma \leq e$. And by means of the Arzelà-Ascoli theorem, $T : E \rightarrow E$, is completely continuous [10].

3. Existence and Uniqueness results

In this section, we prove the uniqueness result via Banach contraction principle.

Theorem 12. Assume that there are $L > 0$ such that

$$|f(\varsigma, \varpi) - f(\varsigma, v)| \leq L \|\varpi - v\|,$$

$$\forall \varpi, v \in \mathbb{R}_+, \varsigma \in [1, e],$$

and if

$$C = L \int_1^e \left(G(e, s) + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(e, s) dt \right) \frac{ds}{s} < 1, \quad \varsigma, s \in [1, e]$$

Then, the problem (1.1), has a unique solution in E

Proof. We will use the Banach contraction principle to prove that the operator T defined by (2.4) has a fixed point. Now we will prove that T is a contraction. Let $\varpi, v \in E$, we get

$$\begin{aligned}
|Tu(\varsigma) - Tv(\varsigma)| &\leq \\
&\int_1^e \left(G(\varsigma, s) |f(s, \varpi(s)) - f(s, v(s))| + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(\varsigma, s) |f(s, \varpi(s)) - f(s, v(s))| dt \right) \frac{ds}{s}
\end{aligned}$$

So, we can obtain

$$\leq L \int_1^e \left(G(e, s) + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(e, s) dt \right) \frac{ds}{s} \|\varpi - v\|_E$$

By using

$$C = L \int_1^e \left(G(e, s) + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(e, s) dt \right) \frac{ds}{s} < 1.$$

Obviously, we have

$$\|Tu - Tv\|_E \leq C \|\varpi - v\|_E,$$

so, the contraction principle ensures the uniqueness of a solution for the fractional boundary value

problem (1.1). This finishes the proof.

The existence results are based on the following Leray-Schauder nonlinear alternative.

Lemma 13. [8] Let F be Banach space and Ω be a bounded open subset of F , $0 \in \Omega$. $T : \overline{\Omega} \rightarrow F$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega$, $\lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$

Theorem 14. Assume that there exist nonnegative functions $k, h \in L^1[1, e]$ such that

$$|f(\varsigma, \varpi)| \leq k(\varsigma)|\varpi| + h(\varsigma), (\varsigma, \varpi) \in [1, e] \times \mathbb{R},$$

and

$$F = \int_1^e \left(G(e, s)k(s) + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(e, s)k(s)dt \right) \frac{ds}{s} < 1,$$

$$G = \int_1^e \left(G(e, s)h(s) + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(e, s)h(s)dt \right) \frac{ds}{s} < 1.$$

Then the BVP (1.1) has at least one solution $\varpi^* \in E$.

Proof. To prove this Theorem, we apply Lemma 13. First, we need to prove that T is completely continuous:

1) $G(\varsigma, s)$ is continuous and $f(\varsigma, \varpi(\varsigma))$ is continuous nonnegative function, we get that $T : E \rightarrow E$, is continuous.

2) Let $B_r = \{\varpi \in E : \|\varpi\|_E \leq r\}$ a bounded subset. we will prove that $T(\Omega \cap B_r)$ is relatively compact:

(i) For some $\varpi \in \Omega \cap B_r$, we have:

$$|Tu(\varsigma)| \leq \int_1^e \left(G(\varsigma, s)|f(s, \varpi(s))| + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(\varsigma, s)|f(s, \varpi(s))|dt \right) \frac{ds}{s}.$$

From the above inequalities we have

$$\begin{aligned} |Tu(\varsigma)| &\leq \|\varpi\| \int_1^e \left(G(e, s)k(s) + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(e, s)k(s)dt \right) \frac{ds}{s} \\ &\quad + \int_1^e \left(G(e, s)h(s) + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(e, s)h(s)dt \right) \frac{ds}{s} \end{aligned}$$

This shows that

$$\|Tu\|_E \leq F\|\varpi\|_E + G = Fm + G.$$

$$\|Tu\|_X \leq F\|\varpi\|_X + G \leq Fr + G,$$

then, $T(\Omega \cap B_r)$ uniformly bounded.

(ii) The continuity of $G(\varsigma, s)$ implies that, for any $\varepsilon > 0$, there exists a constant δ such that $\forall \varsigma_1, \varsigma_2 \in [1, e]$ if $|\varsigma_1 - \varsigma_2| < \delta$, then $|G(\varsigma_1, s) - G(\varsigma_2, s)| < \varepsilon$.

We have:

$$|Tu(\varsigma_1) - Tu(\varsigma_2)| \leq \int_1^e |(G(\varsigma_1, s) - G(\varsigma_2, s)) f(s, \varpi(s))| \\ + \frac{\eta[(\log \varsigma_1)^{q-1} - (\log \varsigma_2)^{q-1}]}{1 - \eta \int_a^e [(\log \varsigma_1)^{q-1} - (\log \varsigma_2)^{q-1}] dt} \int_1^e (G(\varsigma_1, s) - G(\varsigma_2, s)) f(s, \varpi(s)) dt \left| \frac{ds}{s} \right|.$$

So,

$$|Tu(\varsigma_1) - Tu(\varsigma_2)| \leq L \left[\int_1^e |G(\varsigma_1, s) - G(\varsigma_2, s)| \right. \\ \left. + \frac{\eta[(\log \varsigma_1)^{q-1} - (\log \varsigma_2)^{q-1}]}{1 - \eta \int_a^e [(\log \varsigma_1)^{q-1} - (\log \varsigma_2)^{q-1}] dt} \int_1^e |G(\varsigma_1, s) - G(\varsigma_2, s)| dt \frac{ds}{s} \right]$$

where, $L = \max_{a < s < e} |f(s, \varpi(s))|$

As $\varsigma_1 \rightarrow \varsigma_2$, the right-hand side of the above inequality tends to zero, $|Tu(\varsigma_1) - Tu(\varsigma_2)| \xrightarrow{\varsigma_1 \rightarrow \varsigma_2} 0$, consequently $T(\Omega \cap B_r)$ is equicontinuous. From Arzela-Ascoli theorem, we deduce that T is a completely continuous operator.

Now, we prove that there exists a point $\varpi \in E$ which satisfies $Tu = \varpi$.

Consider $\Omega = \{\varpi \in X : \|\varpi\| < m\}$, with $m = G(1 - F)^{-1}$,

$$\lambda m = \lambda \|\varpi\| = \|Tu\|_E.$$

We assume that $\varpi \in \partial\Omega$, $\lambda > 1$ such that $Tu = \lambda u$, then

$$|Tu(\varsigma)| \leq \int_1^e \left(G(e, s) |f(s, \varpi(s))| + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(e, s) |f(s, \varpi(s))| dt \right) \frac{ds}{s}.$$

We also have,

$$|Tu(\varsigma)| \leq \|\varpi\| \int_1^e \left(G(e, s) k(s) + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(e, s) k(s) dt \right) \frac{ds}{s} \\ + \int_1^e \left(G(e, s) h(s) + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(e, s) h(s) dt \right) \frac{ds}{s}$$

This shows that

$$\lambda m = \|Tu\|_E \leq F \|\varpi\|_E + G = Fm + G.$$

From this we get

$$\lambda \leq F + \frac{G}{m} = F + \frac{G}{G(1-F)^{-1}} = F + (1-F) = 1,$$

consequently $\lambda < 1$, this contradicts $\lambda > 1$. By applying *Lemma 13*, T has a fixed point $\varpi^* \in \overline{\Omega}$, which is a solution of the Hadamard fractional boundary value problem (1.1).

The proof is complete.

4. Positivity results

In this section, we discuss the existence of positive solution for the Hadamard *fractional boundary value problem* (1.1). We make the following additional assumptions.

(Q1) $f(\varsigma, \varpi(\varsigma)) = a(\varsigma) f_1(\varpi(\varsigma))$ where $a \in C([1, e], \mathbb{R}_+)$ and $f_1 \in C(\mathbb{R}_+, \mathbb{R}_+)$.

(Q2) $0 < \int_1^e (\log \varsigma)^{q-1} G(e, s) a(s) \frac{ds}{s} < \infty$, $0 < \int_1^e G(e, s) a(s) \frac{ds}{s} < \infty$.

Definition 15. A function $\varpi(\varsigma)$ is called positive solution for the fractional boundary value problem (1.1) if $\varpi(\varsigma) \geq 0$, $\forall \varsigma \in [0, 1]$ and satisfies, the BVP (1.1)

Lemma 16. Let $\varpi \in E$, the unique solution ϖ of the fractional boundary value problem (1.1) is nonnegative and satisfies

$$\min_{\varsigma \in [\tau, e]} \varpi(\varsigma) \geq (\log \tau)^{q-1} \|\varpi\|_E,$$

Proof. Let $\varpi \in E$, it is obvious that $\varpi(\varsigma)$ is nonnegative, $\varsigma \in [a, e]$. From *Lemma 11*, we have

$$\|\varpi\|_E \leq \int_1^e \left(G(e, s) a(s) f_1(\varpi(s)) + \frac{\eta (\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(e, s) a(s) f_1(\varpi(s)) dt \right) \frac{ds}{s},$$

and,

$$\begin{aligned} \varpi(\varsigma) &\geq \int_1^e \left((\log \varsigma)^{q-1} G(e, s) a(s) f_1(\varpi(s)) \right. \\ &\quad \left. + \frac{\eta (\log \varsigma)^{q-1}}{\Lambda} \int_1^e (\log \varsigma)^{q-1} G(e, s) a(s) f_1(\varpi(s)) dt \right) \frac{ds}{s}, \\ &\geq (\log \varsigma)^{q-1} \|\varpi\|_E. \end{aligned}$$

Hence, for all $\varsigma \in [\tau, e]$, we obtain

$$\min_{\varsigma \in [\tau, e]} \varpi(\varsigma) \geq (\log \tau)^{q-1} \|\varpi\|_E.$$

The proof is complete.

Definition 17. We define the cone K by

$$K = \left\{ \varpi \in E, \varpi(\varsigma) \geq 0, \min_{\varsigma \in [\tau, e]} (\varpi(\varsigma)) \geq (\log \tau)^{q-1} \|\varpi\|_E \right\}$$

K is a non-empty closed and convex subset of E .

We define an operator $T : K \rightarrow K$, as follows:

$$Tu(\varsigma) = \int_1^e H(\varsigma, s) a(s) f_1(\varpi(s)) \frac{ds}{s}.$$

$$\begin{aligned} Tu(\varsigma) &= \int_1^e \left(G(\varsigma, s) + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(\varsigma, s) dt \right) a(\varsigma) f_1(\varpi(s)) \frac{ds}{s}, \\ &= \int_1^e \left(G(\varsigma, s) a(\varsigma) f_1(\varpi(s)) + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(\varsigma, s) a(\varsigma) f_1(\varpi(s)) dt \right) \frac{ds}{s}. \end{aligned}$$

The operator T is continuous in view of continuity of $G(\varsigma, s)$, $a(\varsigma)$ and $f_1(\varpi(\varsigma))$. And by means of the Arzelà-Ascoli theorem, T is completely continuous.

By Lemma 11, we have

$$\|Tu\| \leq \int_1^e \left(G(e, s) a(s) f_1(\varpi(s)) + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(e, s) a(s) f_1(\varpi(s)) dt \right) \frac{ds}{s},$$

and,

$$\begin{aligned} Tu(\varsigma) &\geq \int_1^e \left((\log \varsigma)^{q-1} G(e, s) a(s) f_1(\varpi(s)) \right. \\ &\quad \left. + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e (\log \varsigma)^{q-1} G(e, s) a(s) f_1(\varpi(s)) dt \right) \frac{ds}{s}, \\ &\geq (\log \varsigma)^{q-1} \|Tu\|. \end{aligned}$$

Therefore, $T(K) \subset K$

Lemma 18. [10] *The operator defined in (4.1) is completely continuous and satisfies $T(K) \subseteq K$.*

To establish the existence of positive solutions for problem (1.1), we will employ the Guo-Krasnosel'skii fixed point theorem.

The main result of this section is the following:

Theorem 19. *Let (Q_1) and (Q_2) hold, $0 < \beta \eta^{\alpha-1} < 1$ and assume that*

$$f_0 = \lim_{|\varpi| \rightarrow 0} \frac{f_1(\varpi)}{|\varpi|}, \quad f_\infty = \lim_{|\varpi| \rightarrow \infty} \frac{f_1(\varpi)}{|\varpi|} \text{ exist.}$$

Then the problem (1.1) has at least one positive solution in the case

- (i) $f_0 = 0$ and $f_\infty = \infty$ (superlinear) or
- (ii) $f_0 = \infty$ and $f_\infty = 0$ (sublinear).

Proof. We will prove that the problem B.V.P (1.1) has at least one positive solution in both cases, superlinear and sublinear. For this we use Theorem 8. We prove the superlinear case.

Since $f_0 = 0$, then for any $\varepsilon > 0$, $\exists \delta_1 > 0$, such that $f_1(\varpi) \leq \varepsilon |\varpi|$, for $|\varpi| < \delta_1$. Let Ω_1 be an open set in E defined by

$$\Omega_1 = \{y \in E / \|y\|_E < \delta_1\}$$

then, for any $\varpi \in K \cap \partial\Omega_1$, it yields

$$\|Tu\|_E \leq \int_1^e \left(G(e, s)a(s)f(\varpi(s)) + \frac{\mu(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(e, s)a(s)f(\varpi(s))dt \right) \frac{ds}{s},$$

$$\|Tu\|_E \leq \varepsilon \|\varpi\|_E \int_1^e \left(G(e, s)a(s) + \frac{\mu(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(e, s)a(s)dt \right) \frac{ds}{s},$$

If we choose $\varepsilon = \left[\int_1^e \left(G(e, s)a(s) + \frac{\mu(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(e, s)a(s)dt \right) \frac{ds}{s} \right]^{-1}$, then it yields

$$\|Tu\|_E \leq \|\varpi\|_E, \quad \forall \varpi \in K \cap \partial\Omega_1.$$

Now from $f_\infty = \infty$, then $\forall M > 0, \exists H > 0$, such that $f_1(\varpi) \geq M(|\varpi|)$ for $|\varpi| \geq H$. Let

$$H_1 = \max \{2\delta_1, (\log \tau)^{q-1} H\}$$

Denote by Ω_2 the open set

$$\Omega_2 = \{y \in E / \|y\|_E < H_1\}$$

For any $\varpi \in K \cap \partial\Omega_2$, we have

$$\min_{\varsigma \in [\tau, e]} \varpi(\varsigma) \geq (\log \tau)^{q-1} \|\varpi\|_E,$$

$$= (\log \tau)^{q-1} H_1 \geq H,$$

let $\varpi \in K \cap \partial\Omega_2$ then

$$Tu(\varsigma) \geq \int_1^e \left((\log \varsigma)^{q-1} G(e, s)a(s)f(\varpi(s)) + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e (\log \varsigma)^{q-1} G(e, s)a(s)f(\varpi(s))dt \right) \frac{ds}{s},$$

$$Tu(\varsigma) \geq M \int_1^e \left((\log \varsigma)^{q-1} G(e, s)a(s) + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e (\log \varsigma)^{q-1} G(e, s)a(s)dt \right) \frac{ds}{s} \|\varpi\|_E,$$

and choosing $M = \left[\int_1^e \left((\log \varsigma)^{q-1} G(e, s)a(s) + \frac{\eta(\log \varsigma)^{q-1}}{\Lambda} \int_1^e (\log \varsigma)^{q-1} G(e, s)a(s)dt \right) \frac{ds}{s} \right]^{-1}$, we get

$$\|Tu\|_E \geq \|\varpi\|_E, \quad \forall \varpi \in K \cap \partial\Omega_2.$$

By the first part of *Theorem 8*, T has at least one fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$, such that;

$H \leq \|\varpi\| \leq H_1$. This completes the superlinear case of *Theorem 19*. **Case II** Now, we assume that

$f_0 = \infty$ and $f_\infty = 0$ (sublinear case), Proceeding as above and by the second part of *Theorem 8*, we prove the sublinear case. This achieves the proof of *Theorem 19*.

5. Examples

In order to illustrate our result, we give the following examples:

Example 20. Consider the following fractional boundary value problem

$$\begin{cases} {}^H D^{\frac{5}{2}} \varpi(\varsigma) + \frac{1}{2(5+\varpi(\varsigma))\exp(\varsigma+3)} = 0, & \varsigma \in (1, e), \\ \varpi(1) = \varpi'(1) = 0, & x(1) = \frac{3}{2} \int_1^e \varpi(\varsigma) dt, \end{cases}$$

set

$$q = \frac{5}{2}, \eta = \frac{3}{2},$$

and

$$f(\varsigma, \varpi(\varsigma)) = \frac{1}{2(5+\varpi(\varsigma))\exp(\varsigma+3)}, \quad \forall \varpi \in \mathbb{R}, \varsigma \in [1, e].$$

One can choose

$$k(\varsigma) = \frac{1}{2\exp(\varsigma+3)}, \quad \varsigma \in [1, e],$$

$k \in L^1([1, e], \mathbb{R}^+)$ are nonnegative functions, where

$$\begin{aligned} |f(\varsigma, \varpi(\varsigma)) - f(\varsigma, v(\varsigma))| &\leq \frac{1}{2\exp(\varsigma+3)} \frac{|\varpi(\varsigma) - v(\varsigma)|}{|(5+\varpi(\varsigma))(5+v(\varsigma))|} \\ &\leq \frac{1}{2\exp(\varsigma+3)} |\varpi(\varsigma) - v(\varsigma)| \\ &\leq k(\varsigma) |\varpi(\varsigma) - v(\varsigma)|. \end{aligned}$$

and,

$$C = L \int_1^e \left(G(e, s) + \frac{\frac{3}{2}(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(e, s) dt \right) \frac{ds}{s} < 1.$$

Hence, by Theorem 12, the Hadamard fractional boundary value problem (J1) has a unique solution in E .

Example 21. Consider the following fractional boundary value problem

$$\begin{cases} {}^H D^{\frac{5}{2}} \varpi(\varsigma) + (1+\varsigma^2) \varpi(\varsigma) + \exp(-\varsigma) = 0, & \varsigma \in (a, e), \\ \varpi(a) = \varpi'(a) = 0, & x(1) = \frac{3}{2} \int_1^e \varpi(\varsigma) dt, \end{cases}$$

set

$$q = \frac{5}{2}, \eta = \frac{3}{2}.$$

Where,

$$f(\varsigma, \varpi(\varsigma)) = \varpi(\varsigma) \sin \alpha + \exp(-\varsigma).$$

$k, l \in L^1([1, e], \mathbb{R}^+)$ are nonnegative functions, where

$$\begin{aligned} |f(\varsigma, \varpi(\varsigma))| &\leq (1+\varsigma^2) |\varpi(\varsigma)| + \frac{1}{\exp \varsigma}, \quad \forall \varpi \in \mathbb{R}, \varsigma \in [1, e], \\ &\leq k(\varsigma) |\varpi(\varsigma)| + l(\varsigma), \end{aligned}$$

and

$$F = \int_1^e \left(G(e, s)k(s) + \frac{\frac{3}{2}(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(e, s)k(s)dt \right) \frac{ds}{s} < 1,$$

$$G = \int_1^e \left(G(e, s)h(s) + \frac{\frac{3}{2}(\log \varsigma)^{q-1}}{\Lambda} \int_1^e G(e, s)h(s)dt \right) \frac{ds}{s} < 1.$$

Hence, by *Theorem 14*, the Hadamard fractional boundary value problem (J2) has at least one solution in E .

Example 22. Consider the following fractional boundary value problem

$$\begin{cases} {}^H D^{\frac{5}{2}} \varpi(\varsigma) + \varpi^2(\varsigma) \ln \varsigma = 0, & \varsigma \in (a, e), \\ \varpi(1) = \varpi'(1) = 0, & x(1) = \frac{3}{2} \int_1^e \varpi(\varsigma) dt, \end{cases}$$

where,

$$q = \frac{5}{2}, \quad \eta = \frac{3}{2},$$

and

$$f(\varsigma, \varpi(\varsigma)) = \varpi^2(\varsigma) \ln \varsigma = a(\varsigma) f_1(\varpi(\varsigma)),$$

$$a(\varsigma) = \ln \varsigma \in C([1, e], \mathbb{R}_+), \quad f_1(\varpi) \in C(\mathbb{R}_+, \mathbb{R}_+). \quad \text{Then}$$

$$f_0 = \lim_{|\varpi| \rightarrow 0} \frac{f_1(\varpi)}{|\varpi|} = 0, \quad \text{and} \quad f_\infty = \lim_{|\varpi| \rightarrow \infty} \frac{f_1(\varpi)}{|\varpi|} = \infty.$$

By *Theorem 19 (i)*, the fractional boundary value problem (J3) has at least one positive solution.

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