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Analytical solutions to the differential equations of surface density evolution in protoplanetary disks

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Abstract

This study provides analytical solutions to the differential equations governing the evolution of surface density in protplanetary disks, with particular emphasis on how this evolution influences the dynamical behavior of planets within the disk. A viscous, axisymmetric disk model is adopted to analyse the time evolution of surface density and its connection to planetary motion and angular momentum transport. The results showed good agreement with earlier models, strengthening the mathematical framework for understanding of protoplanetary disk dynamics and clarifying both their evolution and their impact on the motion of forming planets.

Keywords: Hankel Transform, Weber Transform, Protoplanetary Disk, Surface Density, Angular Momentum.

1. INTRODUCTION

The formation and evolution of protoplanetary disks are fundamental processes that govern planet formation and the early dynamical evolution of planetary systems. These disks are primarily composed of gas and dust and exhibit highly complex behaviors influenced by viscosity, self-gravity, pressure gradients, and interactions with embedded planets. Understanding the distribution and temporal evolution of surface density is crucial, as it directly affects angular momentum transport, planetary migration, accretion rates, and the overall dynamical stability of the system [1] [6] [11] [10].

Analytical solutions to the governing differential equations play a vital role in complementing numerical simulations. They allow researchers to study the temporal and spatial evolution of disk properties under well-defined initial conditions and boundary constraints, providing a systematic framework for understanding disk-planet interactions [2] [13] [14].

In this context, integral transforms such as the Hankel transform and Weber transform serve as powerful mathematical tools for addressing the partial differential equations that arise in axisymmetric disk systems. The Hankel transform is particularly suited for analyzing radial dependencies in circularly symmetric systems, converting radial differential equations into simpler forms that can be solved analytically. Conversely, the Weber transform is applied to cases with specific boundary conditions, enabling accurate solutions in situations where conventional methods may fail [3] [12].

The use of these transforms significantly simplifies the mathematical complexity associated with viscous disk evolution and angular momentum transport. By applying the Hankel and Weber transforms, researchers can obtain accurate analytical solutions that reveal how surface density redistributes over time, how angular momentum is transferred throughout the disk, and how these processes affect planetary migration and dynamical evolution. These analytical approaches also provide a robust framework for validating numerical models and interpreting observational data of protoplanetary disks [4] [16].

In this study, a viscous, axisymmetric disk model was employed to investigate the evolution of surface density and its impact on planetary motion and angular momentum transfer. By focusing on the application of Hankel and Weber transforms as primary mathematical tools, we aimed to obtain accurate analytical solutions that not only reproduce the known behavior of viscous disks but also offer new insights into disk dynamics. The results show good agreement with previous analytical and numerical models, reinforcing the theoretical understanding of disk-planet interactions and clarifying the mechanisms governing the orbital behavior of forming planets within evolving disks [5] [17].

The goal of this research is to find exact time-dependent solutions to the protoplanetary disk equation based on angular momentum behavior. This is what distinguishes our work from previous research . This work is presented as follows. Section contains the evolution equations of the protoplanetary disk as well as the angular momentum in order to calculate the surface density of the disk, we also show the solution to the evolution equation based on the behavior of angular momentum. The final section presents the most important results obtained with the overall conclusion of this work.

2. Surface density evolution equation

We present models of giant planet migration in developing protoplanetary disks, which eventually lead to the central star's transmission of viscous angular momentum. Planets migrate in the Type II migration regime as a result of tidal interaction with the disk [18] [19], and the disk is also subject to tidal torques from planets, according to equation for the coupled evolution of a protoplanetary disk and planet [6] [15].

$$\frac{\partial\Sigma}{\partial t} = \frac{1}{R}\frac{\partial}{\partial R}\left[3R^{1/2}\frac{\partial}{\partial R}\left(\eta\Sigma R^{1/2}\right) - \frac{2\Lambda\Sigma R^{3/2}}{(GM)^{1/2}}\right]. \qquad (1)$$

where $\Sigma(\mathrm{R},\mathrm{t})$ is the disk surface density,t is time, R is cylindrical radius, η is the kinematic viscosity, and M is the stellar mass. The first term on the righthand side describes ordinary viscous evolution of the disk [3],[7] and The second term describe show the disk responds to the planetary torque. Here Λ is the injection rate of angular momentum per unit mass into the disk. Following [8],[9] for a planet of mass $\mathrm{M}_p = \mathrm{qM}$ in circular orbit at radius a, the torque distribution has the form.

$$\Lambda(R) = \begin{cases} \dfrac{q^2GM}{2R}\left(\dfrac{R}{\Delta p}\right)^2 if\ \ R > \alpha \\ \dfrac{-q^2GM}{2R}\left(\dfrac{R}{\Delta p}\right)^4 if\ \ R < \alpha \end{cases} . \qquad (2)$$

In our model, we will solve the equation of the evolution equation (1) and find an analytical expression of the torque without resorting to numerical calculation. By assuming a separable ansatz of the form $\Sigma(\mathrm{R},\mathrm{t}) = \varphi(\mathrm{R})\exp(-\lambda(\mathrm{t}))$, where λ is real number and $\varphi(\mathrm{R})$ is an arbitrary function of R. The equation (1) transforms, after the substitution indicated above, to the following differential equation.

$$0 = R^2\varphi''(R) + \left(2n + \frac{3}{2} - \frac{2}{\sqrt{GM^*\eta}}\Lambda R^{3/2}\right)R\varphi'(R) + \left[n^2\frac{n}{2}\frac{2}{\sqrt{GM^*\eta}}\Lambda' R^{5/2} - \frac{3}{\sqrt{GM^*\eta}}\Lambda R^{3/2} + \frac{\lambda}{3\eta}R^2\right]\varphi(R)\ . \qquad (3)$$

By making the change of variable R to $(a\mathrm{x})$, $\varphi(\mathrm{R})$ changes to $\mathrm{u}(\mathrm{x})$ $\eta(\mathrm{R})$ to $\nu(\mathrm{x})$ we find the following equation that governs $\mathrm{u}(\mathrm{x})$

$$0=\frac{u''(x)}{u(x)}+\left(\frac{3}{2x}+2\frac{v'(x)}{v(x)}-\frac{2\sqrt{x}}{3v(x)}l(x)\right)\frac{u'(x)}{u(x)}$$
$$+\left(\frac{3}{2x}\frac{v'(x)}{v(x)}+\frac{v''(x)}{v(x)}-\frac{l(x)}{\sqrt{x}v(x)}-\frac{2\sqrt{x}}{3v(x)}l'(x)+\frac{\lambda}{3v(x)}\right), \quad (4)$$

where:

$$l(x)=\frac{\Lambda\alpha^{3/2}}{\sqrt{GM}} \quad ; \quad l'(x)=\frac{\Lambda\alpha^{3/2}}{\sqrt{GM}}\frac{d\Lambda}{dx} \quad .(5)$$

the equation (4) has a generic form

$$0=u''(x)+f_1(x)u'(x)+f_0(x)u(x) \quad , \quad (6)$$

where $f_0(x)$ and $f_1(x)$ are respectively given by

$$f_0(x)=\frac{3}{2x}\frac{v'(x)}{v(x)}+\frac{v''(x)}{v(x)}-\frac{l(x)}{\sqrt{x}v(x)}-\frac{2\sqrt{x}}{3v(x)}l'(x)+\frac{\lambda}{3v(x)} \quad , \quad (7)$$

$$f_1(x)=\frac{3}{2x}+2\frac{v'(x)}{v(x)}-\frac{2\sqrt{x}}{3v(x)}l(x) \quad . \ (8)$$

In order to solve the equation (6) we put the change

$$u(x)=\frac{\omega(x)}{q(x)} \quad . \ (9)$$

After simplifications, it is traightforward and easy to get

$$\frac{\omega''(x)}{\omega(x)}+\left(f_1(x)-2\frac{q'(x)}{q(x)}\right)\frac{\omega'(x)}{\omega(x)}+\left[f_0(x)-\frac{q''(x)}{q(x)}+2\left(\frac{q'(x)}{q(x)}\right)^2-f_1(x)\frac{q'(x)}{q(x)}\right]=0\,, \quad (10)$$

by using

$$f_1(x)-2\frac{q'(x)}{q(x)}=\frac{g}{x} \quad , \quad (11)$$

$$G(x)=f_0(x)-\frac{q''(x)}{q(x)}+2\left(\frac{q'(x)}{q(x)}\right)^2-f_1(x)\frac{q'(x)}{q(x)} \quad , \quad (12)$$

where g is a constant, then the equation (11)(12) transforms to

$$0 = \omega''(x) + \frac{g}{x}\omega'(x) + G(x)\omega(x) \quad , \qquad (13)$$

we will use for our convenience later and or equivalently, after using the formula (12), (13)

we find

$$G(x) = f_0(x) - \frac{f_1'(x)}{2} - \frac{f_1^2(x)}{4} + \frac{g(g-2)}{4x^2} \quad , \qquad (14)$$

we substitute the expressions of $f_0(x)$ and $f_1(x)$and given above, we fin

$$G(x) = \frac{3}{16x^2} + \frac{\lambda}{3v(x)} - \frac{l(x)}{3\sqrt{x}v(x)} - \frac{\sqrt{x}}{3v(x)}\frac{d}{dx}l(x) - \frac{x}{9v^2(x)}l^2(x) + \frac{\sqrt{x}v'(x)}{3v(x)}l(x) + \frac{g(g-2)}{4x^2}, \qquad (15)$$

replace the reduced viscosity $\nu(x)$ by Sx^{β}, then:

$$\frac{\omega''(x)}{\omega(x)} + G(x) - 2\frac{q'(x)}{q(x)} + \frac{\omega'(x)}{\omega(x)} + \frac{\lambda}{3Sx^{\beta}} + \frac{g(g-2)}{4x^2} - \frac{1}{3sx^{\beta-\frac{1}{2}}}\frac{d}{dx}l(x) - \frac{1}{9s^2x^{2\beta+\frac{1}{2}}}l(x) + \frac{\beta-1}{3sx^{\beta+\frac{1}{2}}}l(x) \quad , \quad (16)$$

to solve the above differential equation (16), that is to say:

$$0 = \frac{d}{dx}l(x) + \frac{l^2(x)}{3Sx^{\beta-1/2}} - \frac{(\beta-1)}{x}l(x) \quad , \quad (17)$$

so

$$l(x) = \frac{3Sx^{\beta-1}}{3cs+2\sqrt{x}} \quad , \quad (18)$$

and

$$G(x) = \frac{3}{16x^2} + \frac{\lambda}{3Sx^{\beta}} + \frac{g(g-2)}{4x^2}, \quad (19)$$

the differential equation that governs $\omega(x)$

$$0 = \omega''(x) + \frac{g}{x}\omega'(x) + \left(\frac{3}{16x^2} + \frac{\lambda}{3Sx^{\beta}} + \frac{g(g-2)}{4x^2}\right)\omega(x) \quad , \quad (20)$$

the viscosity $v(x) = Sx^{\beta}$, $\frac{\lambda}{3S} = k^2$

$$\omega''(x) + \frac{g}{x}\omega'(x) + \left(\frac{\lambda}{3Sx^{\beta}} + \frac{3+4g(g-2)}{16x^2}\right)\omega(x) = 0 \ , \quad (21)$$

$$x^2\omega''(x) + gx\omega'(x) + \left(k^2x^{2-B} + \frac{3+4g(g-2)}{16}\right)\omega(x) = 0 \ , \ (22)$$

bessel equation whose general solution is

$$\omega(x) = x^{\frac{1}{2}(1-g)}[C_1(k)J_v(ky) + C_2(k)Y_v(ky)] \ , \quad (23)$$

where $y = \frac{1}{1-B/2}x^{1-B/2}$, $v = \frac{1}{2(2-B)}$ and $J_v(ky)$ and $Y_v(ky)$ are the well known Bessel's functions

$$u(x) = x^{(1-g)/2}\exp\left(-\frac{1}{2}\int^{x} f_1(z)\,dz + \frac{g}{2}ln(x)\right)\left(C_1(k)J_v(ky) + C_2(k)Y_v(ky)\right), (24)$$

or after replacing f_1 by its expression and the torque $l(x)$ by its expression, we find

$$u(x) = \frac{x^{-\frac{5}{4}}}{3CS}l^{-1}(x)x^{\frac{1}{2}(1-g)}[C_1(k)J_v(ky) + C_2(k)Y_v(ky)] \ , (25)$$

such

the general solution is the summation over all possible modes k, that is to say

$$\Sigma(x,t) = \int_0^{\infty}\frac{x^{-\frac{5}{4}}}{3CS}l^{-1}(x)x^{\frac{1}{2}(1-g)}[C_1(k)J_v(ky) + C_2(k)Y_v(ky)]exp(-3Sk^2t)dk. \quad (26)$$

We will clarify the above solution (26) in each region ($x < 1$ and $x > 1$) by giving appropriate parameters in each region. This is the subject of the following two subsections.

2.1 In the regionI $R < a, x < 1$

asymptotic behavior

$$\Sigma(x,t) = \int_0^\infty \frac{x^{-\frac{5}{4}}}{3CS} l^{-1}(x) x^{\frac{1}{2}(1-g)} C_1(k) J_v(ky) exp(-3Sk^2t) dk \quad , (27)$$

$$\Sigma(x,0) = \frac{x^{-\frac{5}{4}}}{3CS} l^{-1}(x) x^{\frac{1}{2}(1-g)} \int_0^\infty C_1(k) J_v(ky) \ , \ (28)$$

to avoid a possible source of divergence at both the origin ($x = 0$ in region 1) and at ($x = \infty$ in region 2), we put $g = -3/2$, then

$$3CSl(x) x^{\frac{5}{4}} x^{\frac{1}{2}(g-1)} \Sigma(x,0) = \int_0^\infty C_1(k) k^{-1} J_v(ky) k dk \ , \ (29)$$

hankel transforms

$$C_1(k) k^{-1} = \int_0^\infty 3CSl(x) x^{\frac{5}{4}} x^{\frac{1}{2}(g-1)} \Sigma(x,0) J_v(ky) y dy \quad , \quad (30)$$

$$C_1(k) = \int_0^\infty 3CSl(x) x^{\frac{5}{4}} x^{\frac{1}{2}(g-1)} \Sigma(x,0) J_v(ky) y dy \quad , \qquad (31)$$

where $y = \frac{1}{1-B/2} x^{1-B/2}$,

we can find

$$C_1(k) = (1 - B/2)^{-1} \int_0^1 3CSl(x) x^{\frac{5}{4}} x^{\frac{1}{2}(g-1)} \Sigma(x,0) J_v(ky) k x^{1-B} dx \ , \qquad (32)$$

we can write

$$C_1(k) = (1 - B/2)^{-1} \int_0^1 3CSl(x') x'^{\frac{5}{4}} x'^{\frac{1}{2}(g-1)} \Sigma(y',0) J_v(ky') k x'^{1-B} dx' \ , \qquad (33)$$

but

$$\Sigma(x,t) = \int_0^\infty \frac{x^{-\frac{5}{4}}}{3CS} l^{-1}(x) x^{\frac{1}{2}(1-g)} C_1(k) J_v(ky) exp(-3Sk^2t) dk \ , \ (34)$$

$$\Sigma(x,t) = \left(1 \ -\frac{B}{2}\right)^{-1} \int_0^1 \frac{x^{-\frac{5}{4}}}{3CS} l^{-1}(x) x^{\frac{1}{2}(1-g)}$$

.

$$\times \int_0^\infty 3CSl(x') x'^{\frac{5}{4}} x'^{\frac{1}{2}(g-1)} \Sigma(y',0) J_v(ky') k x'^{1-B} dx' \, J_v(ky) exp(-3Sk^2 t) dk \ , \quad (35)$$

$$\Sigma(x,t) = \left(1 - \frac{B}{2}\right)^{-1} x^{-\frac{5}{4}} l^{-1}(x) x^{-\frac{1}{2}(g-1)} \int_0^1 l(x') x'^{\frac{5}{4}} x'^{\frac{1}{2}(g-1)} x'^{1-B}$$

:

$$\times \int_0^\infty \Sigma(y',0) J_v(ky') J_v(ky) exp(-3Sk^2 t) k dk dx' \ , \quad (36)$$

but

$$\Sigma(x,t) = \int_0^{y(1)} G(x,x',t) \Sigma(x',0) \, dx' \quad , \qquad (37)$$

So

$$G(x,x',t) = \left(1 - \frac{B}{2}\right)^{-1} x^{-\frac{5}{4}} l^{-1}(x) x^{\frac{1}{2}(g-1)} l(x') x'^{\frac{5}{4}} x'^{\frac{1}{2}(g-1)} x'^{1-B}$$

$$\int_0^\infty J_v(ky') J_v(ky) exp(-3Sk^2 t) k dk \quad , \ (38)$$

$$\int_0^\infty J_v(ky') J_v(ky) exp(-3Sk^2 t) k dk = \frac{1}{6St} exp\left(-\frac{y^2 + y'^2}{12St}\right) I_p\left(\frac{yy'}{6St}\right) \quad , (39)$$

$$\int_0^\infty J_v(ky') J_v(ky) exp(-3Sk^2 t) k dk = \frac{1}{6St} exp\left(-\frac{1 + (x'/x)^{2-B}}{12St(1-B/2)^2 x^{B-2}}\right) I_p\left(\frac{2(x'/x)^{2-B/2}}{12St(1-B/2)^2 x^{B-2}}\right) \quad , \qquad (40)$$

$$G(x,x',t) = \left(1 - \frac{B}{2}\right)^{-1} x^{-\frac{5}{4}} l^{-1}(x) x^{\frac{1}{2}(g-1)} l(x') x'^{\frac{5}{4}} x'^{\frac{1}{2}(g-1)} x'^{1-B}$$

$$. \frac{1}{6St} exp\left(-\frac{1 + (x'/x)^{2-B}}{12St(1-B/2)^2 x^{B-2}}\right) I_p\left(\frac{2(x'/x)^{2-B/2}}{12St(1-B/2)^2 x^{B-2}}\right) \quad , \ (41)$$

but

$$\Sigma(x,t) = \int_0^{y(1)} G(x,x',t) \Sigma(x',0) \, dx' \quad , \ (42)$$

and

$$\Sigma(x',0) = \Sigma_0 \delta(x' - 1/2) \quad , \qquad (43)$$

$$\Sigma(x,\tau) = \Sigma_0 2 \left(-\frac{B}{2}\right)^{-1} (2x)^{-\frac{5}{4}} (2x)^{\frac{1}{2}(g-1)} \frac{q_1 +.(2x)^{\frac{1}{2}}}{q_1 + 1} (2x)^{1-B} (x)^{B-2} \left(\frac{1}{2}\right)^{1-B}$$

$$.\frac{1}{\tau} exp\left(-\frac{1+(2x)^{\frac{B}{2}-1}}{\tau}\right) I_p \left(\frac{2(2x)^{\frac{B}{2}-1}}{\tau}\right) \quad ,(44)$$

As $\tau = 12St(1 - B/2)^2 x^{B-2}, \frac{l(1/2)}{l(x)} = \frac{q_1 +.(2x)^{\frac{1}{2}}}{q_1 + 1}.(2x)^{1-B}, \; q_1 = \frac{3CS}{\sqrt{2}},$

we can plot the density as a function of x at values of $B < 2$, at different times τ, g, q_1 constant values

$$q_1 = \frac{3CS}{\sqrt{2}} = -1.41 \quad , \quad B = \frac{3}{2} \quad ; \quad g = -\frac{3}{2} \quad , \qquad (45)$$

so

$$\frac{\Sigma(x,\tau)}{W_0} = \frac{1-\sqrt{x}}{x}.\frac{1}{\tau} exp\left(-\frac{1+(2x)^{\frac{B}{2}-1}}{\tau}\right) I_p \left(\frac{2(2x)^{\frac{B}{2}-1}}{\tau}\right) \quad ,(46)$$

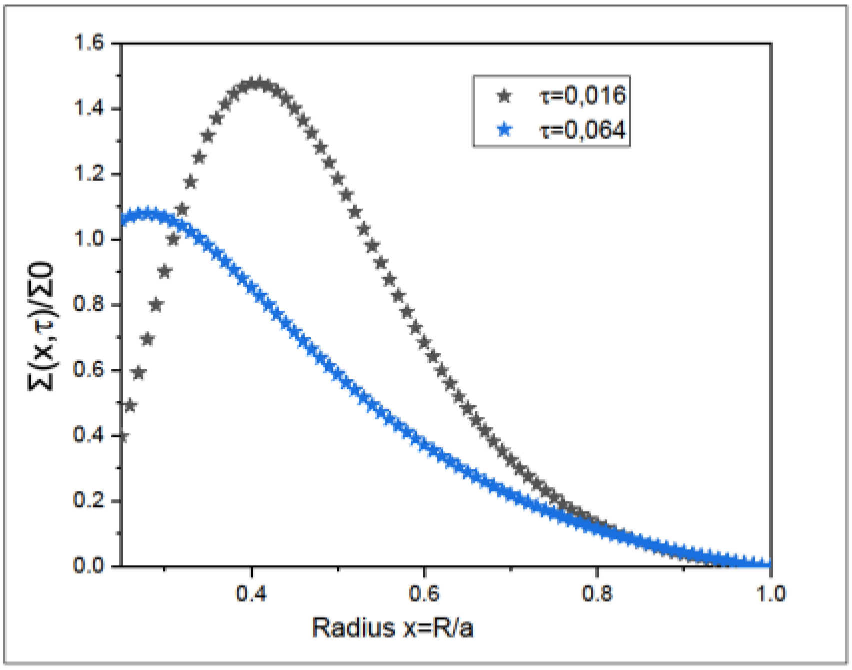

Figure 1: surface density in the regionI

2.2 In the regionII $R > a, x > 1$

Rewrite the general equation

on prend les modes de k

$$\Sigma(x,t) = \int_0^\infty \frac{x^{-\frac{5}{4}}}{3CS} l^{-1}(x) x^{\frac{1}{2}(1-g)} [C_1(k)J_v(ky) + C_2(k)Y_v(ky)] exp(-3Sk^2t)dk \,, \qquad (47)$$

$$\Sigma(x,t) = \int_0^\infty F(x)[C_1(k)J_v(ky) + C_2(k)Y_v(ky)] exp(-3Sk^2t)dk \qquad , \qquad (48)$$

$$\Sigma(x,t) = \int_0^\infty F(x)[C_1(k)J_v(ky) + C_2(k)Y_v(ky)] exp(-3Sk^2t)dk \qquad , \qquad (49)$$

$$\frac{\Sigma(x,t)}{F(x)} = \int_0^\infty [C_1(k)J_v(ky) + C_2(k)Y_v(ky)] exp(-3Sk^2t)dk \qquad , \qquad (50)$$

$$F(x) = \frac{x^{-\frac{5}{4}}}{3CS} l^{-1}(x) x^{\frac{1}{2}(1-g)} \quad , \quad (51)$$

Using limit at $x = 1$ $\Sigma(x = 1, t) = 0$ then

$$\left[C_1(k)J_v\big(ky(1)\big) + C_2(k)Y_v\big(ky(1)\big)\right] = 0 \quad , \quad (52)$$

$$C_2(k) = -\frac{J_v\big(ky(1)\big)}{Y_v\big(ky(1)\big)} \qquad , (53)$$

then

$$\frac{\Sigma(x,t)}{F(x)} = \int_0^\infty \frac{C_1(k)}{Y_v\big(ky(1)\big)} \left[Y_v\big(ky(1)\big)J_v\big(ky(x)\big) - J_v\big(ky(1)\big)Y_v\big(ky(x)\big)\right] exp(-3Sk^2t)dk \;, \quad (54)$$

$$\frac{\Sigma(x,t)}{F(x)} = \int_0^\infty \frac{C(k)}{k} \left[Y_v\big(ky(1)\big)J_v\big(ky(x)\big) - J_v\big(ky(1)\big)Y_v\big(ky(x)\big)\right] exp(-3Sk^2t)dk \quad , (55)$$

$$\frac{\Sigma(x,0)}{F(x)} = \int_0^\infty \frac{C(k)}{k} \left[Y_v\big(ky(1)\big)J_v\big(ky(x)\big) - J_v\big(ky(1)\big)Y_v\big(ky(x)\big)\right] kdk \quad , (56)$$

$$\frac{\Sigma(x,0)}{F(x)} = \int_0^\infty \frac{C(k)\left(J_v{}^2\big(ky(1)\big) + Y_v{}^2\big(ky(1)\big)\right)}{k} \frac{\left[Y_v\big(ky(1)\big)J_v\big(ky(x)\big) - J_v\big(ky(1)\big)Y_v\big(ky(x)\big)\right]}{J_v{}^2\big(ky(1)\big) + Y_v{}^2\big(ky(1)\big)} kdk \quad , (57)$$

the transformation for weber

$$\frac{C(k)\left(J_v{}^2\big(ky(1)\big)+Y_v{}^2\big(ky(1)\big)\right)}{k}$$

$$= \int_0^\infty \frac{\Sigma(x',0)}{F(x')}\left[Y_v\big(ky(1)\big)J_v\big(ky(x')\big) - J_v\big(ky(1)\big)Y_v\big(ky(x')\big)\right]x'dx' \qquad (58)$$

$$\frac{C(k)}{k} = \int_1^\infty \frac{\Sigma(x',t)}{F(x')}\frac{\left[Y_v\big(ky(1)\big)J_v\big(ky(x')\big) - J_v\big(ky(1)\big)Y_v\big(ky(x')\big)\right]}{J_v{}^2\big(ky(1)\big)+Y_v{}^2\big(ky(1)\big)}x'dx' \quad , \qquad (59)$$

$$\frac{\Sigma(x,t)}{\Sigma_0} = \int_1^\infty \frac{F(x)}{F\left(\frac{3}{2}\right)}\left[\frac{Y_v\big(ky(1)\big)J_v\left(ky\left(\frac{3}{2}\right)\right) - J_v\big(ky(1)\big)Y_v\left(ky\left(\frac{3}{2}\right)\right)}{J_v{}^2\big(ky(1)\big)+Y_v{}^2\big(ky(1)\big)}\right]\frac{3}{2}.\left[Y_v\big(ky(1)\big)J_v\big(ky(x)\big)\right.$$
$$\left.- J_v\big(ky(1)\big)Y_v\big(ky(x)\big)\right]exp(-3Sk^2t)kdk \quad , \qquad (60)$$

where

$$F\left(\frac{3}{2}\right) = \frac{\frac{3}{2}^{-\frac{5}{4}}}{3CS}l^{-1}\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)^{\frac{1}{2}(1-g)} \quad ,(61)$$

$$F(x) = \frac{x^{-\frac{5}{4}}}{3CS}l^{-1}(x)(x)^{\frac{1}{2}(1-g)} \quad ,(62)$$

$$\frac{F(x)}{F\left(\frac{3}{2}\right)} = \left(\frac{2x}{3}\right)^{-\frac{5}{4}}\frac{l\left(\frac{3}{2}\right)}{l(x)}\left(\frac{2x}{3}\right)^{\frac{1}{2}(1-g)} = \left(\frac{2x}{3}\right)^{-\frac{5}{4}}\frac{\frac{q_1}{\sqrt{3}}+\sqrt{\frac{2x}{3}}}{\frac{q_1}{\sqrt{3}}+1}\left(\frac{2x}{3}\right)^{1-B}\left(\frac{2x}{3}\right)^{\frac{1}{2}(1-g)} \quad , \qquad (63)$$

$$\frac{\Sigma(x,t)}{W_0} = \left(\frac{\sqrt{x}-1}{\sqrt{x}}\right)\int_1^\infty \left[\frac{Y_v\big(ky(1)\big)J_v\left(ky\left(\frac{3}{2}\right)\right) - J_v\big(ky(1)\big)Y_v\left(ky\left(\frac{3}{2}\right)\right)}{J_v{}^2\big(ky(1)\big)+Y_v{}^2\big(ky(1)\big)}\right]\left[Y_v\big(ky(1)\big)J_v\big(ky(x)\big)\right.$$
$$\left.- J_v\big(ky(1)\big)Y_v\big(ky(x)\big)\right]exp(-3Sk^2t)kdk \quad , \quad (64)$$

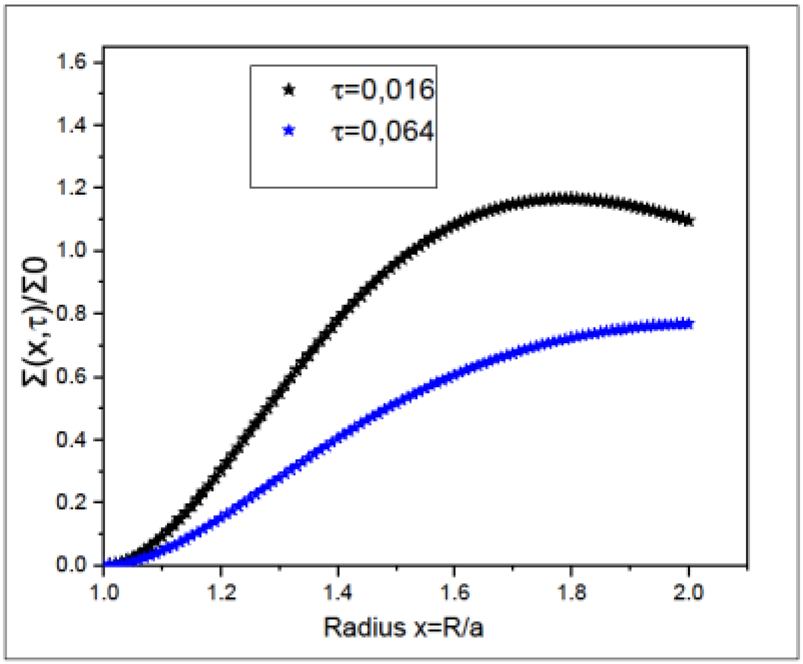

Figure 2: The surface density in region

3. Results and Conclusion

The general approach includes studying the evolution of the planet in the protoplanetary disk. Dynamic interactions with stellar companions and planet formation are among the many processes responsible for the formation of protoplanetary disks, and the gravitational interaction between the disk and the planet results in the exchange of angular momentum and planet migration. In this work, we describe the dynamics of the gravitational interaction between the planet and the protoplanetary disk, and we show how the evolution equations can be remeasured to a form similar to previous simulations. Fig.1 and Fig.2 show how surface density changes near and far from the planet.

In this paper, we describe the dynamics of a planet's gravitational interaction with a protoplanetary disk. We also present preliminary findings from our research

We assumed that in the region ($x < 1$) and at the initial time, the surface density increases at the site ($x = 1/2$), and that as time passes, the surface density decreases and declines near the planet (Fig.1). When the density of the protoplanetary disk surface decreases, the planet creates a large disk gap as its mass increases and becomes gigantic and massive. The viscosity is also low in this case, and a standard migration of (the second type) occurs.Our results are consistent with an approximation of [6].

In the region ($x > 1$), as we have assumed, the surface density behavior away from the planet is greater and more pronounced at position ($x = 3/2$). In this case, the viscosity is large, and the gap becomes less exhaustive and less resistant to gas, resulting in a separation of migration with respect to disk development (Fig.2), We demonstrated in both regions how planetary torque affects the surface density of the disk by finding an analytical solution to the evolution equation of the disk and anembedded protoplanet. And the results were satisfactory when compared to the previous results [6].

In this paper, we have obtained an overview of gaseous proto-planetary disk and embedded planet interactions. We found similar solutions in previous studies. What distinguishes our work is the solu tion of the surface density equation for the protoplanetary disk without using an angular momentum approximation. What the figures show in both regions demonstrates the validity of our work.

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