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Stability Properties of Solutions of Certain Systems of Difference Equations Involving Tribonacci Numbers

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Abstract:

In this work we interested the form of solutions, stability character and asymptotic of the system non-linear of difference equations in order $p+1$.

$$\begin{cases} x_{n+1} = \frac{\pm 1}{y_{n-(p-1)}(x_{n-p} \pm 1)+1}, \\ y_{n+1} = \frac{\pm 1}{x_{n-(p-1)}(y_{n-p} \pm 1)+1} \end{cases} \quad n \in \mathbb{N}_0, \quad p \geq 1$$

where $x_{-p}, x_{-(p-1)}, x_{-(p-2)}, \ldots, x_0, y_{-p}, y_{-(p-1)}, y_{-(p-2)}, \ldots, y_0$ are real initial values with the certain conditions.

Keywords : Tribonacci numbers, Stability, Equilibrium point, Solutions, System of Difference equations.

0.1 Introduction

Difference equations usually describe the evolution of certain phenomena over the course of time [1], The theory of difference equations developed greatly during the last twenty-five years of the twentieth century. Also the difference equation is the relation between the terms of the unknown sequence, which obtained from known principles, and then the equation is solved by a mathematical or numerical methods.

Its applications is rapidly increasing to various fields such as numerical analysis, biology, economics, control theory, finite mathematics and computer science.

In 2013 [11], Y. Yazlik, D.T.Tollu and N.Taskara are investigated the form of the solutions of the following rational difference equation systems

$$\begin{cases} x_{n+1} = \frac{x_{n-1} \pm 1}{y_n x_{n-1}}, \\ y_{n+1} = \frac{y_{n-1} \pm 1}{x_n y_{n-1}} \end{cases} \quad n = 0, 1, 2, \ldots \tag{1}$$

such that their solutions are associated with Padovan numbers.

In 2016 [2], Y. Halim concerned with the periodicity and the stability of the solutions of the system of difference equations

$$\begin{cases} x_{n+1} = \frac{1}{1+y_{n-2}}, \\ y_{n+1} = \frac{1}{1+x_{n-2}} \end{cases} \quad n \in \mathbb{N}_0 \tag{2}$$

In 2021 [10], I. Talha, S.Badidja has studied the periodicity of solutions of the following general system rational of difference equations

$$\begin{cases} x_{n+1} = \frac{y_n(x_{n-(p-1)}+y_{n-p})}{y_{n-p}+x_{n-(p-1)}-y_n}, \\ y_{n+1} = \frac{x_{n-(p-2)}(x_{n-(p-2)}+y_{n-(p-1)})}{2x_{n-(p-2)}+y_{n-(p-1)}}, \end{cases} \quad n = 0, 1, 2, \ldots \tag{3}$$

which $p = 2, 3, ...$
In 2019 [9], İ. Okumuş, Y. Soykan are study the explicit form, stability character and global behavior of solutions of the following two systems of rational difference equations

$$\begin{cases} x_{n+1} = \frac{1}{y_{n-(p-1)}(x_{n-p}+1)+1}, \\ y_{n+1} = \frac{1}{x_{n-(p-1)}(y_{n-p}+1)+1} \end{cases} \quad n \in \mathbb{N}_0, \quad p \geq 1 \tag{4}$$

$$\begin{cases} x_{n+1} = \frac{-1}{y_{n-(p-1)}(x_{n-p}-1)+1}, \\ y_{n+1} = \frac{-1}{x_{n-(p-1)}(y_{n-p}-1)+1} \end{cases} \quad n \in \mathbb{N}_0, \quad p \geq 1 \tag{5}$$

Then, in this work we investigate with the solutions, stability character and assymptotic for the following systems of difference equations

$$\begin{cases} x_{n+1} = \frac{1}{y_{n-(p-1)}(x_{n-p}+1)+1}, \\ y_{n+1} = \frac{1}{x_{n-(p-1)}(y_{n-p}+1)+1} \end{cases} \quad n \in \mathbb{N}_0, \quad p \geq 1 \tag{6}$$

$$\begin{cases} x_{n+1} = \frac{-1}{y_{n-(p-1)}(x_{n-p}-1)+1}, \\ y_{n+1} = \frac{-1}{x_{n-(p-1)}(y_{n-p}-1)+1} \end{cases} \quad n \in \mathbb{N}_0, \quad p \geq 1 \tag{7}$$

0.2 Preliminaries

Tribonacci numbers :

Definition 0.2.1 *[7] The Tribonacci numbers T_n are defined by the recurrence linear relation*

$$\begin{cases} T_n = T_{n-1} + T_{n-2} + T_{n-3}, n \geq 4 \\ T_1 = T_2 = 1 \\ T_3 = 2 \end{cases} \tag{8}$$

the first Tibonacci numbers are 1, 1, 2, 4,7, 13, 24, 44, 81.... *which the characteristic equation associated to the previous recurrence relation is given by :*

$$x^3 - x^2 - x - 1 = 0 \tag{9}$$

it's roots are as follows : α (real root), β and γ (tow complex roots are conjugated).

More generally, we can give the following limit

$$\lim_{n \longrightarrow +\infty} \frac{T_{n+r}}{T_n} = \alpha^r, \quad r \in \mathbb{Z}.$$

0.2.1 Equilibrium point [4]

Now, in the rest of this section we shall present some basic notations and results on the study of nonlinear difference equations which will be useful in our investigation. Let f, g be two continuously differentiable functions :

$$f : I^{p+1} \times J^{p+1} \longrightarrow I, \quad g : I^{p+1} \times J^{p+1} \longrightarrow J$$

with $I, J \subseteq \mathbb{R}$, and for $n \in \mathbb{N}$, we have the system of difference equations

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-p}, y_n, y_{n-1}, \dots, y_{n-p}) \\ \\ y_{n+1} = g(x_n, x_{n-1}, \dots, x_{n-p}, y_n, y_{n-1}, \dots, y_{n-p}). \end{cases} \tag{10}$$

such that $x_{-p}, x_{-(p-1)}, x_{-(p-2)}, \dots, x_0, y_{-p}, y_{-(p-1)}, y_{-(p-2)}, \dots, y_0 \in I^{p+1} \times J^{p+1}$

The map H is defined by

$$H : I^{p+1} \times J^{p+1} \longrightarrow I \times J$$

where

$$H(W) = (f(W), f_1(W), f_2(W), f_3(W), \dots, f_p(W), g(W), g_1(W), g_2(W), g_3(W), \dots, g_p(W))$$

in which

$$W = (u_0, u_1, u_2, \dots, u_p v_0, v_1, v_2, \dots, v_p)^T$$

,

$$f_1(W) = u_0, \quad f_2(W) = u_1, \quad f_p(W) = u_{p-1}$$
$$g_1(W) = v_0, \quad g_2(W) = v_1, \quad g_p(W) = v_{p-1}$$

Let

$$W_n = (x_n, x_{n-1}, \dots, x_{n-p}, y_n, y_{n-1}, \dots, y_{n-p})^T$$

it's noted that the system (10) is equivalent to the following radial system

$$W_{n+1} = H(W_n), \quad n = 0, 1, 2, ..., \tag{11}$$

Which

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, x_{n-2}, x_{n-3}, \dots, x_{n-p}, y_n, y_{n-1}, y_{n-2}, y_{n-3} \dots, y_{n-p}) \\ x_n = x_n \\ x_{n-1} = x_{n-1} \\ x_{n-2} = x_{n-2} \\ \quad \dots\dots\dots \\ \quad \dots\dots\dots \\ \quad \dots\dots\dots \\ x_{n-(p-1)} = x_{n-(p-1)} \\ y_{n+1} = g(x_n, x_{n-1}, x_{n-2}, x_{n-3}, \dots, x_{n-p}, y_n, y_{n-1}, y_{n-2}, y_{n-3} \dots, y_{n-p}) \\ y_n = y_n \\ y_{n-1} = y_{n-1} \\ y_{n-2} = y_{n-2} \\ \quad \dots\dots\dots \\ \quad \dots\dots\dots \\ \quad \dots\dots\dots \\ y_{n-(p-1)} = y_{n-(p-1)}. \end{cases}$$

The notion of equilibrium points is central in the study of the dynamics of any physical system. In many applications in biology, economics, physics, etc. It's desirable that all solutions of a given system tend to it's equilibrium point. This is the subject of study of stability theory.

Now we provide definition for equilibrium point :

Definition 0.2.2 *(Equilibrium point)*[4] *The equilibrium point* $(\bar{x}, \bar{y}) \in I \times J$ *of the system* (10) *is a solution of the system :*

$$\begin{cases} x = f(x, x, \dots, x, y, y, \dots, y) \\ y = g(x, x, \dots, x, y, y, \dots, y) \end{cases}$$

Also, an equilibrium point $\overline{W} \in I^{p+1} \times J^{p+1}$ *of system* (11) *is a solution of the system*

$$W = H(W).$$

Definition 0.2.3 *(Stability)*[2] *Let* $\overline{W}$ *be an equilibrium point of system* (11) *and* $\|.\|$ *be any norm (e.g., the Euclidean norm).*

1. *The equilibrium point* $\overline{W}$ *is called stable (or locally stable) if for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that* $\|W_0 - \overline{W}\| < \delta$ *implies* $\|W_n - \overline{W}\| < \epsilon, \quad$ *for* $n \geq 0$.
2. *The equilibrium point* $\overline{W}$ *is called asymptotically stable (or locally asymptotically stable) if it is stable and there exists* $\gamma > 0$ *such that* $\|W_0 - \overline{W}\| < \gamma$ *implies*

$$\|W_n - \overline{W}\| \longrightarrow 0, \quad n \longrightarrow +\infty.$$

3. *The equilibrium point* $\overline{W}$ *is said to be global attractor (respectively global attractor with basin of attraction a set* $G \subseteq I^{p+1} \times J^{p+1}$*), if for every* W_0 *(respectively for every* $W_0 \in G$*)*

$$\|W_n - \overline{W}\| \longrightarrow 0, \quad n \longrightarrow +\infty.$$

4. *The equilibrium point* $\overline{W}$ *is called globally asymptotically stable (respectively globally asymptotically stable relative to G) if it is asymptotically stable, and if for every* W_0 *(respectively for every* $W_0 \in G$*)*

$$\|W_n - \overline{W}\| \longrightarrow 0, \quad n \longrightarrow +\infty.$$

5. *The equilibrium point* $\overline{W}$ *is called unstable if it is not stable.*

Theoreme 0.2.1 *(Rouche's theorem)* ([1], [8]) *Let* $f(\lambda), g(\lambda)$ *two holomorphic functions in an open* Ω *of the complexe plane* $\mathbb{C}$*, and either* K *a compact contained in* Ω*, with*

$$| g(\lambda) | < | f(\lambda) |, \quad \forall \lambda \in \partial K.$$

Then the number of zeros of $f(\lambda) + g(\lambda)$ *in* K *is equal to the number of zeros of* $f(\lambda)$ *in* K.

Remarque 0.2.1 $(\bar{x}, \bar{y}) \in I \times J$ *is an equilibrium point for system* (10) *if and only if* $\overline{W} = (\bar{x}, \bar{x}, \dots, \bar{x}\bar{y}, \bar{y}, \dots, \bar{y}) \in I^{p+1} \times J^{p+1}$ *is an equilibrium point of system* (11).
The linearized system associated to the system (11) *according to the equilibrium point* $\overline{W} = (\bar{x}, \bar{x}, \dots, \bar{x}\bar{y}, \bar{y}, \dots, \bar{y}) \in I^{p+1} \times J^{p+1}$ *is given by*

$$W_{n+1} = MW_n, n = 0, 1, ...$$

where M *is the Jacobian matrix of the application about the equilibrium point* $\overline{W}$ *given by*

$$A = \begin{bmatrix} \frac{\partial f}{\partial u_0}(\overline{W}) & \dots & \frac{\partial f}{\partial u_p}(\overline{W}) & \frac{\partial f}{\partial v_0}(\overline{W}) & \dots & \frac{\partial f}{\partial v_p}(\overline{W}) \\ \frac{\partial f_1}{\partial u_0}((\overline{W})) & \dots & \frac{\partial f_1}{\partial u_k}(\overline{W}) & \frac{\partial f_1}{\partial v_0}(\overline{W}) & \dots & \frac{\partial f_1}{\partial v_k}(\overline{W}) \\ \vdots & \dots & \ddots & \dots & \ddots & \vdots \\ \vdots & \dots & \ddots & \dots & \ddots & \vdots \\ \frac{\partial f_p}{\partial u_0}(\overline{W}) & \dots & \frac{\partial f_p}{\partial u_p}(\overline{W}) & \frac{\partial f_p}{\partial v_0}(\overline{W}) & \dots & \frac{\partial f_p}{\partial v_p}(\overline{W}) \\ \frac{\partial g}{\partial u_0}((\overline{W})) & \dots & \frac{\partial g_0}{\partial u_p}(\overline{W}) & \frac{\partial g_0}{\partial v_0}(\overline{W}) & \dots & \frac{\partial g_0}{\partial v_p}(\overline{W}) \\ \vdots & \dots & \ddots & \dots & \ddots & \vdots \\ \vdots & \dots & \ddots & \dots & \ddots & \vdots \\ \frac{\partial g_p}{\partial u_0}(\overline{W}) & \dots & \frac{\partial g_p}{\partial u_p}(\overline{W}) & \frac{\partial g_p}{\partial v_0}(\overline{W}) & \dots & \frac{\partial g_p}{\partial v_p}(\overline{W}) \end{bmatrix}$$

Theoreme 0.2.2 [2] *If all the eigenvalues of the Jacobian matrix M lie in the open unit disk $|\lambda| < 1$, then the equilibrium point $\overline{W}$ of system (11) is asymptotically stable.*
On the other hand, if at least one eigenvalue of the Jacobian matrix M have absolute value greater than one, then the equilibrium point $\overline{W}$ of system (11) is unstable.

0.3 Main Result

In this paper we deal the form of solutions of the rational difference equations systems (6) and (7).

0.3.1 First system

In the first part, we interested to the form of solutions, then studied the stability of the following system of rational difference equations

$$\begin{cases} x_{n+1} = \frac{1}{y_{n-(p-1)}(x_{n-p}+1)+1}, \\ y_{n+1} = \frac{1}{x_{n-(p-1)}(y_{n-p}+1)+1} \end{cases} \quad n \in \mathbb{N}_0, \quad p \geq 1 \tag{12}$$

which the initial conditions of the negative index terms :
$x_{-p}, x_{-(p-1)}, x_{-(p-2)}, \ldots, x_0, y_{-p}, y_{-(p-1)}, y_{-(p-2)}, \ldots, y_0 \in \mathbb{R} - F$
with

$$F = \cup \left\{(x_{-p}, x_{-(p-1)}, x_{-(p-2)}, \ldots, x_0, y_{-p}, y_{-(p-1)}, y_{-(p-2)}, \ldots, y_0) : A_n = 0, B_n = 0, C_n = 0, D_n = 0\right\}$$

such that

$$\begin{aligned} A_n &= T_{2n-(2p-1)} x_{i-(p+1)} y_{p-i} + (T_{2n-2p} + T_{2n-(2p-1)}) y_{p-i} + T_{2n-(2p-2)} \\ B_n &= T_{2n-(2p-1)} y_{i-(p+1)} x_{p-i} + (T_{2n-2p} + T_{2n-(2p-1)}) x_{p-i} + T_{2n-(2p-2)} \\ C_n &= T_{2n-(2p-2)} y_{p-1} x_{i-(p+1)} + (T_{2n-(2p-1)} + T_{2n-(2p-2)}) x_{i-(p+1)} + T_{2n-(2p-3)} \\ D_n &= T_{2n-(2p-2)} x_{p-i} y_{i-(p+1)} + (T_{2n-(2p-1)} + T_{2n-(2p-2)}) y_{i-(p+1)} + T_{2n-(2p-3)}. \end{aligned}$$

The following theorem describes the form of solutions of system (12).

Theoreme 0.3.1 *Let $\{x_n, y_n\}_{n\geq 0}$ be the solutions of system (12). Then for $n = 0, 1, 2, ...$; and $p \geq 1$, the forms of $\{x_n, y_n\}_{n\geq 0}$ are given as*
1/ *For $i = 1, 2, \ldots, p$*

$$x_{2pn-i} = \frac{T_{2n-2p} x_{i-(p+1)} y_{p-i} + (T_{2n-(2p-2)} - T_{2n-(2p-1)}) y_{p-i} + T_{2n-(2p-1)}}{T_{2n-(2p-1)} x_{i-(p+1)} y_{p-i} + (T_{2n-2p} + T_{2n-(2p-1)}) y_{p-i} + T_{2n-(2p-2)}}, \tag{13}$$

2/ *For $i = 1, 2, \ldots, p$*

$$y_{2pn-i} = \frac{T_{2n-2p} y_{i-(p+1)} x_{p-i} + (T_{2n-(2p-2)} - T_{2n-(2p-1)}) x_{p-i} + T_{2n-(2p-1)}}{T_{2n-(2p-1)} y_{i-(p+1)} x_{p-i} + (T_{2n-2p} + T_{2n-(2p-1)}) x_{p-i} + T_{2n-(2p-2)}}, \tag{14}$$

3/ *For $i = p+1, p+3, \ldots, 2p$*

$$x_{2pn-(i-1)} = \frac{T_{2n-(2p-1)} y_{p-i} x_{i-(p+1)} + (T_{2n-(2p-3)} - T_{2n-(2p-2)}) x_{i-(p+1)} + T_{2n-(2p-2)}}{T_{2n-(2p-2)} y_{p-i} x_{i-(p+1)} + (T_{2n-(2p-1)} + T_{2n-(2p-2)}) x_{i-(p+1)} + T_{2n-(2p-3)}}, \tag{15}$$

4/ For $i = p+1, p+3, \ldots, 2p$

$$y_{2pn-(i-1)} = \frac{T_{2n-(2p-1)}x_{p-i}y_{i-(p+1)} + (T_{2n-(2p-3)} - T_{2n-(2p-2)})y_{i-(p+1)} + T_{2n-(2p-2)}}{T_{2n-(2p-2)}x_{p-i}y_{i-(p+1)} + (T_{2n-(2p-1)} + T_{2n-(2p-2)})y_{i-(p+1)} + T_{2n-(2p-3)}}, \tag{16}$$

where T_n the nth Tribonacci numbers.

Proof

By induction

The result hold for $k=0$. Now we suppose that $k>0$ and that our assumption holds for $k-1$. That is

For $i = 1, 2, \ldots, p$

$$x_{2p(k-1)-i} = \frac{T_{2k-(2p+1)}y_{i-(p+1)}x_{p-i} + (T_{2k-(2p-1)} - T_{2k-2p})x_{p-i} + T_{2k-2p}}{T_{2k-2p}y_{i-(p+1)}x_{p-i} + (T_{2k-(2p+1)} + T_{2k-2p})x_{p-i} + T_{2k-(2p-1)}}, \tag{17}$$

For $i = 1, 2, \ldots, p$

$$y_{2p(k-1)-i} = \frac{T_{2k-(2p+1)}x_{i-(p+2)}y_{i-(p+1)} + (T_{2k-(2p-1)} - T_{2k-2p})y_{i-(p+1)} + T_{2k-2p}}{T_{2k-2p}x_{i-(p+2)}y_{i-(p+1)} + (T_{2k-(2p+1)} + T_{2k-2p})y_{i-(p+1)} + T_{2k-(2p-1)}}, \tag{18}$$

For $i = p+1, p+2, \ldots, 2p$

$$x_{2p(k-1)-(i-1)} = \frac{T_{2k-(2p+2)}x_{i-(2p+2)}y_{i-(2p+1)} + (T_{2k-2p} - T_{2k-(2p+1)})y_{i-(2p+1)} + T_{2k-(2p+1)}}{T_{2k-(2p+1)}x_{i-(2p+2)}y_{i-(2p+1)} + (T_{2k-(2p+2)} + T_{2k-(2p+1)})y_{i-(2p+1)} + T_{2k-2p}}, \tag{19}$$

For $i = p+1, p+2, \ldots, 2p$

$$y_{2p(k-1)-(i-1)} = \frac{T_{2k-(2p+2)}y_{i-(2p+2)}x_{i-(2p+1)} + (T_{2k-2p} - T_{2k-(2p+1)})x_{i-(2p+1)} + T_{2k-(2p+1)}}{T_{2k-(2p+1)}y_{i-(2p+2)}x_{i-(2p+1)} + (T_{2k-(2p+2)} + T_{2k-(2p+1)})x_{i-(2p+1)} + T_{2k-2p}}, \tag{20}$$

In the other hand and for $i = 1, 2, 3, \ldots, p$, it follows from (12), (17) and (18)

$$\begin{aligned}
x_{2pk-i} &= \frac{1}{y_{2p(k-1)-i}(x_{2p(k-1)-(i-1)}+1)+1} \\
&= \frac{1}{\frac{T_{2k-(2p+1)}x_{i-(p+1)}y_{p-i} + (T_{2k-(2p-1)} - T_{2k-2p})y_{p-i} + T_{2k-2p}}{T_{2k-2p}x_{i-(p+1)}y_{p-i} + (T_{2k-(2p+1)} - T_{2k-2p})y_{p-i} + T_{2k-(2p-1)}} \times \left(x_{2p(k-1)-(i-1)}+1\right)+1} \\
&= \frac{1}{y_{2k-2p} \times \left(\frac{T_{2k-(2p+2)}x_{i-(p+1)}y_{p-i} + (T_{2k-2p} - T_{2k-(2p+1)})y_{p-i} + T_{2k-(2p+1)}}{T_{2k-(2p+1)}x_{i-(p+1)}y_{p-i} + (T_{2k-(2p+2)} - T_{2k-(2p+1)})y_{p-i} + T_{2k-2p}}+1\right)+1} \\
&= \frac{T_{2k-2p}x_{i-(p+1)}y_{p-i} + (T_{2k-(2p-2)} - T_{2k-(2p-1)})y_{p-i} + T_{2k-(2p-1)}}{T_{2k-(2p-1)}x_{i-(p+1)}y_{p-i} + (T_{2k-2p} + T_{2k-(2p-1)})y_{p-i} + T_{2k-(2p-2)}}
\end{aligned}$$

Then

$$x_{2pk-i} = \frac{T_{2k-2p}x_{i-(p+1)}y_{p-i} + (T_{2k-(2p-2)} - T_{2k-(2p-1)})y_{p-i} + T_{2k-(2p-1)}}{T_{2k-(2p-1)}x_{i-(p+1)}y_{p-i} + (T_{2k-2p} + T_{2k-(2p-1)})y_{p-i} + T_{2k-(2p-2)}}$$

and

$$\begin{aligned}
y_{2pk-i} &= \frac{1}{x_{2p(k-1)-i}(y_{2p(k-1)-(i-1)}+1)+1} \\
&= \frac{1}{\frac{T_{2k-(2p+1)}y_{i-(p+1)}x_{p-i} + (T_{2k-(2p-1)} - T_{2k-2p})x_{p-i} + T_{2k-2p}}{T_{2k-2p}y_{i-(p+1)}x_{p-i} + (T_{2k-(2p+1)} + T_{2k-2p})x_{p-i} + T_{2k-(2p-1)}} \times \left(y_{2p(k-1)-(i-1)}+1\right)+1} \\
&= \frac{1}{x_{2k-2p} \times \left(\frac{T_{2k-(2p+2)}y_{i-(p+1)}x_{p-i} + (T_{2k-2p} - T_{2k-(2p+1)})x_{p-i} + T_{2k-(2p+1)}}{T_{2k-(2p+1)}y_{i-(p+1)}x_{p-i} + (T_{2k-(2p+2)} + T_{2k-(2p+1)})x_{p-i} + T_{2k-2p}}+1\right)+1} \\
&= \frac{T_{2k-2p}y_{i-(p+1)}x_{p-i} + (T_{2k-(2p-2)} - T_{2k-(2p-1)})x_{p-i} + T_{2k-(2p-1)}}{T_{2k-(2p-1)}y_{i-(p+1)}x_{p-i} + (T_{2k-2p} + T_{2k-(2p-1)})x_{p-i} + T_{2k-(2p-2)}}
\end{aligned}$$

Hence,

$$y_{2pk-i} = \frac{T_{2k-2p}y_{i-(p+1)}x_{p-i} + (T_{2k-(2p-2)} - T_{2k-(2p-1)})x_{p-i} + T_{2k-(2p-1)}}{T_{2k-(2p-1)}y_{i-(p+1)}x_{p-i} + (T_{2k-2p} + T_{2k-(2p-1)})x_{p-i} + T_{2k-(2p-2)}}$$

Similarly, for $i = p+1, p+2, \ldots, 2p$ and from (12), (19) and (20) we obtaind

$$\begin{aligned}
x_{2pk-(i-1)} &= \frac{1}{y_{2pk-i}(x_{2p(k-1)-i}+1)+1} \\
&= \frac{1}{\frac{T_{2k-2p}y_{p-i}x_{i-(p+1)} + (T_{2k-(2p-2)} - T_{2k-(2p-1)})x_{i-(p+1)} + T_{2k-(2p-1)}}{T_{2k-(2p-1)}y_{p-i}x_{i-(p+1)} + (T_{2k-2p} + T_{2k-(2p-1)})x_{i-(p+1)} + T_{2k-(2p-2)}} \times \left(x_{2p(k-1)-i}+1\right)+1} \\
&= \frac{1}{y_{2k-(2p-1)} \times \left(\frac{T_{2k-(2p+1)}y_{p-i}x_{i-(p+1)} + (T_{2k-(2p-1)} - T_{2k-2p})x_{i-(p+1)} + T_{2k-2p}}{T_{2k-2p}y_{p-i}x_{i-(p+1)} + (T_{2k-(2p+1)} + T_{2k-2p})x_{i-(p+1)} + T_{2k-(2p-1)}}+1\right)+1} \\
&= \frac{T_{2k-(2p-1)}y_{p-i}x_{i-(p+1)} + (T_{2k-(2p-3)} - T_{2k-(2p-2)})x_{i-(p+1)} + T_{2k-(2p-2)}}{T_{2k-(2p-2)}y_{p-i}x_{i-(p+1)} + (T_{2k-(2p-1)} + T_{2k-(2p-2)})x_{i-(p+1)} + T_{2k-(2p-3)}}
\end{aligned}$$

then

$$x_{2pk-(i-1)} = \frac{T_{2k-(2p-1)}y_{p-i}x_{i-(p+1)} + (T_{2k-(2p-3)} - T_{2k-(2p-2)})x_{i-(p+1)} + T_{2k-(2p-2)}}{T_{2k-(2p-2)}y_{p-i}x_{i-(p+1)} + (T_{2k-(2p-1)} + T_{2k-(2p-2)})x_{i-(p+1)} + T_{2k-(2p-3)}}$$

and

$$\begin{aligned}
y_{2pk-(i-1)} &= \frac{1}{x_{2pk-i}(y_{2p(k-1)-i}+1)+1} \\
&= \frac{1}{\frac{T_{2k-2p}x_{p-i}y_{i-(p+1)} + (T_{2k-(2p-2)} - T_{2k-(2p-1)})y_{i-(p+1)} + T_{2k-(2p-1)}}{T_{2k-(2p-1)}x_{p-i}y_{i-(p+1)} + (T_{2k-2p} + T_{2k-(2p-1)})y_{i-(p+1)} + T_{2k-(2p-2)}} \times \left(y_{2p(k-1)-i}+1\right)+1} \\
&= \frac{1}{x_{2k-(2p-1)} \times \left(\frac{T_{2k-(2p+1)}x_{p-i}y_{i-(p+1)} + (T_{2k-(2p-1)} - T_{2k-2p})y_{i-(p+1)} + T_{2k-2p}}{T_{2k-2p}x_{p-i}y_{i-(p+1)} + (T_{2k-(2p+1)} + T_{2k-2p})y_{i-(p+1)} + T_{2k-(2p-1)}}+1\right)+1} \\
&= \frac{T_{2k-(2p-1)}x_{p-i}y_{i-(p+1)} + (T_{2k-(2p-3)} - T_{2k-(2p-2)})y_{i-(p+1)} + T_{2k-(2p-2)}}{T_{2k-(2p-2)}x_{p-i}y_{i-(p+1)} + (T_{2k-(2p-1)} + T_{2k-(2p-2)})y_{i-(p+1)} + T_{2k-(2p-3)}}
\end{aligned}$$

So

$$y_{2pk-(i-1)} = \frac{T_{2k-(2p-1)}x_{p-i}y_{i-(p+1)} + (T_{2k-(2p-3)} - T_{2k-(2p-2)})y_{i-(p+1)} + T_{2k-(2p-2)}}{T_{2k-(2p-2)}x_{p-i}y_{i-(p+1)} + (T_{2k-(2p-1)} + T_{2k-(2p-2)})y_{i-(p+1)} + T_{2k-(2p-3)}}$$

Existence of equilibrium point

We have the following system of equations

$$\begin{cases} \bar{x} = \frac{1}{\bar{y}(\bar{x}+1)+1} \\ \\ \bar{y} = \frac{1}{\bar{x}(\bar{y}+1)+1} \end{cases} \tag{21}$$

In (21), by subtracting the second equation from the first equation and by some computations, we get

$$\bar{x} - \bar{y} = \frac{1}{1+\bar{x}\bar{y}+\bar{y}} - \frac{1}{1+\bar{y}\bar{x}+\bar{x}}$$

and by some operations we get the result

$$(\bar{x}-\bar{y})[(1+\bar{x}\bar{y}+\bar{y})(1+\bar{y}\bar{x}+\bar{x})-1]=0$$

if $(1+\bar{x}\bar{y}+\bar{y})(1+\bar{y}\bar{x}+\bar{x})=1$ the equation (21) cannot be satisfied car :

$$\bar{x}=\frac{1}{1+\bar{x}\bar{y}+\bar{y}}=\frac{1+\bar{y}\bar{x}+\bar{x}}{(1+\bar{x}\bar{y}+\bar{y})(1+\bar{y}\bar{x}+\bar{x})}$$

if

$$(1+\bar{x}\bar{y}+\bar{y})(1+\bar{y}\bar{x}+\bar{x})=1$$

we get

$$\bar{x}\neq(1+\bar{y}\bar{x}+\bar{x})$$

So

$$\bar{x}=\bar{y}$$

Then, the system (21) can be written as

$$\begin{cases}\bar{x}^3+\bar{x}^2+\bar{x}-1=0\\ \bar{y}^3+\bar{y}^2+\bar{y}-1=0\end{cases}\tag{22}$$

and the characteristic equation $\bar{x}^3+\bar{x}^2+\bar{x}-1=0$ is having three roots a, b and c where,

$$\begin{cases} a=\frac{1+\sqrt[3]{19+3\sqrt{33}}+\sqrt[3]{19-3\sqrt{33}}}{3}\\ b=\frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega^2\sqrt[3]{19-3\sqrt{33}}}{3}\\ c=\frac{1+\omega\sqrt[3]{19+3\sqrt{33}}+\omega\sqrt[3]{19-3\sqrt{33}}}{3}\end{cases}$$

and $\omega=\frac{-1+i\sqrt{3}}{2}$ is a primitive cube root of unity. Hence the unique real positive equilibrium point of system (12) is given by
$W=(a,a,\ldots,a,a,a,\ldots,a)\in I^{p+1}\times J^{p+1}$. Which a is the real root of the characteristic equation : x^3+x^2+x-1.

Theoreme 0.3.2 *The equilibrium point of system (12) is locally asymptotically stable.*
Proof
Let $I=J=(0,+\infty)$ and consider the functions :

$$f:I^{p+1}\times J^{p+1}\longrightarrow I,\quad g:I^{p+1}\times J^{p+1}\longrightarrow J$$

defined by

$$f(x_n,x_{n-1},\ldots,x_{n-p},y_n,y_{n-1},\ldots,y_{n-p})=\frac{1}{y_{n-(p-1)}(x_{n-p}+1)+1}.$$
$$g(x_n,x_{n-1},\ldots,x_{n-p},y_n,y_{n-1},\ldots,y_{n-p})=\frac{1}{x_{n-(p-1)}(y_{n-p}+1)+1}$$

and we have the following transformation

$$(x_n,x_{n-1},\ldots,x_{n-(p-1)},y_n,y_{n-1},\ldots,y_{n-(p-1)})=(f_1,f_2,\ldots,f_p,g_1,g_2,\ldots,g_p)$$

with

$$\begin{cases} f(x_n, x_{n-1}, \ldots, x_{n-p}, y_n, y_{n-1}, \ldots, y_{n-p}) = \frac{1}{y_{n-(p-1)}(x_{n-p}+1)+1} \\ f_1(x_n, x_{n-1}, \ldots, x_{n-p}, y_n, y_{n-1}, \ldots, y_{n-p}) = x_n \\ f_2(x_n, x_{n-1}, \ldots, x_{n-p}, y_n, y_{n-1}, \ldots, y_{n-p}) = x_{n-1} \\ f_3(x_n, x_{n-1}, \ldots, x_{n-p}, y_n, y_{n-1}, \ldots, y_{n-p}) = x_{n-2} \\ \quad \ldots\ldots\ldots \\ \quad \ldots\ldots\ldots \\ \quad \ldots\ldots\ldots \\ f_p(x_n, x_{n-1}, \ldots, x_{n-p}, y_n, y_{n-1}, \ldots, y_{n-p}) = x_{n-(p-1)} \\ g(x_n, x_{n-1}, \ldots, x_{n-p}, y_n, y_{n-1}, \ldots, y_{n-p}) = \frac{1}{x_{n-(p-1)}(y_{n-p}+1)+1} \\ g_1(x_n, x_{n-1}, \ldots, x_{n-p}, y_n, y_{n-1}, \ldots, y_{n-p}) = y_n \\ g_2(x_n, x_{n-1}, \ldots, x_{n-p}, y_n, y_{n-1}, \ldots, y_{n-p}) = y_{n-1} \\ g_3(x_n, x_{n-1}, \ldots, x_{n-p}, y_n, y_{n-1}, \ldots, y_{n-p}) = y_{n-2} \\ \quad \ldots\ldots\ldots \\ \quad \ldots\ldots\ldots \\ \quad \ldots\ldots\ldots \\ g_p(x_n, x_{n-1}, \ldots, x_{n-p}, y_n, y_{n-1}, \ldots, y_{n-p}) = y_{n-(p-1)}. \end{cases}$$

The linearised system associated to the non-linear system (12) about the positive equilibrium point $W = (a, a, \ldots, a, a, a, \ldots, a) \in I^{p+1} \times J^{p+1}$ *is given by*

$$W_{n+1} = MW_n, \tag{23}$$

which

$$W_n = (x_n, x_{n-1}, x_{n-2}, x_{n-3}, \ldots, x_{n-p}, y_n, y_{n-1}, y_{n-2}, y_{n-3} \ldots, y_{n-p})^T.$$

and M *is* $2p \times 2p$ *Jacobian matrix given as*

$$M = \begin{bmatrix} 0 & 0 & \ldots & 0 & \frac{-a}{(a(a+1)+1)^2} & 0 & 0 & \ldots & \frac{-(a+1)}{(a(a+1)+1)^2} & 0 \\ 1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \ldots & \ddots & \ddots & \ddots & \ddots & \ldots & \vdots & \vdots \\ \vdots & \ddots & \ldots & \ddots & \ddots & \ddots & \ddots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & \frac{-(a+1)}{(a(a+1)+1)^2} & 0 & 0 & 0 & \ldots & 0 & \frac{-a}{(a(a+1)+1)^2} \\ 0 & 0 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \ldots & \ddots & \ddots & \ddots & \ddots & \ldots & \vdots & \vdots \\ \vdots & \ddots & \ldots & \ddots & \ddots & \ddots & \ddots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix}$$

Hence, the characteristic equation of the Jacobian matrix M *is given as*

$$(\lambda^2 + (a-1)\lambda + a^3)(\lambda^2 - (a-1)\lambda + a^3).$$

Numerically we get

$$|\lambda_1| = |\lambda_2| = |\lambda_3| = \ldots = |\lambda_{2p}| \simeq 0.40089 < 1$$

Therfore, the equilibrium point W *is locally asymptotically stable.*

Theoreme 0.3.3 *The equilibrium point of system (12)is globally asymptotically stable.*

Proof

Let $\{x_n, y_n\}_{n>0}$ *be a solution of system (12).*

By definition 0.2.3 we just need only to prove that W *is global attractor, that is*

$$\lim_{n\to+\infty}(x_n, y_n) = W.$$

From theorem 0.3.1, We have

$$\begin{aligned}
\lim_{n\longrightarrow+\infty} x_{2pn-i} &= \lim_{n\longrightarrow+\infty}\frac{T_{2n-2p}x_{i-(p+2)}y_{i-(p+1)} + (T_{2n-(2p-2)} - T_{2n-(2p-1)})y_{i-(p+1)} + T_{2n-(2p-1)}}{T_{2n-(2p-1)}x_{i-(p+2)}y_{i-(p+1)} + (T_{2n-2p} + T_{2n-(2p-1)})y_{i-(p+1)} + T_{2n-(2p-2)}}\\
&= \lim_{n\longrightarrow+\infty}\frac{T_{2n-2p}(x_{i-(p+2)}y_{i-(p+1)} + (\frac{T_{2n-(2p-2)}}{T_{2n-2p}} - \frac{T_{2n-(2p-1)}}{T_{2n-2p}})y_{i-(p+1)} + \frac{T_{2n-(2p-1)}}{T_{2n-2p}})}{T_{2n-(2p-1)}(x_{i-(p+2)}y_{i-(p+1)} + (\frac{T_{2n-2p}}{T_{2n-(2p-1)}} + \frac{T_{2n-(2p-1)}}{T_{2n-(2p-1)}})y_{i-(p+1)} + \frac{T_{2n-(2p-2)}}{T_{2n-(2p-1)}})}\\
&= \frac{x_{i-(p+2)}y_{i-(p+1)} + (\alpha^2-\alpha)y_{i-(p+1)} + \alpha}{x_{i-(p+2)}y_{i-(p+1)} + (\frac{1}{\alpha}+1)y_{i-(p+1)} + \alpha}\left(\lim_{n\longrightarrow+\infty}\frac{T_{2n-2p}}{T_{2n-(2p-1)}}\right)\\
&= \frac{1}{\alpha}\\
&= a.
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\longrightarrow+\infty} & x_{2pn-(i-1)}\\
= & \lim_{n\longrightarrow+\infty}\frac{T_{2n-(2p-1)}y_{i-(2p+2)}x_{i-(2p+1)} + (T_{2n-(2p-3)} - T_{2n-(2p-2)})x_{i-(2p+1)} + T_{2n-(2p-2)}}{T_{2n-(2p-2)}y_{i-(2p+2)}x_{i-(2p+1)} + (T_{2n-(2p-1)} + T_{2n-(2p-2)})x_{i-(2p+1)} + T_{2n-(2p-3)}}\\
= & \lim_{n\longrightarrow+\infty}\frac{T_{2n-(2p-1)}(y_{i-(2p+2)}x_{i-(2p+1)} + (\frac{T_{2n-(2p-3)}}{T_{2n-(2p-1)}} - \frac{T_{2n-(2p-2)}}{T_{2n-(2p-1)}})x_{i-(2p+1)} + \frac{T_{2n-(2p-2)}}{T_{2n-(2p-1)}})}{T_{2n-(2p-2)}(y_{i-(2p+2)}x_{i-(2p+1)} + (\frac{T_{2n-(2p-1)}}{T_{2n-(2p-2)}} + \frac{T_{2n-(2p-2)}}{T_{2n-(2p-2)}})x_{i-(2p+1)} + T_{2n-(2p-3)})}\\
= & \frac{y_{i-(2p+2)}x_{i-(2p+1)} + (\alpha^2-\alpha)x_{i-(2p+1)} + \alpha}{y_{i-(2p+2)}x_{i-(2p+1)} + (\frac{1}{\alpha}+1)x_{i-(2p+1)} + \alpha}\left(\lim_{n\longrightarrow+\infty}\frac{T_{2n-(2p-1)}}{T_{2n-(2p-2)}}\right)\\
= & \frac{1}{\alpha}\\
= & a.
\end{aligned}$$

Similarly we obtained

$$\lim_{n\longrightarrow+\infty} y_{2pn-i} = a, \quad \lim_{n\longrightarrow+\infty} y_{2pn-(i-1)} = a$$

So

$$\lim_{n\to+\infty}(x_n, y_n) = W.$$

Hence, the equilibrium point W *is globally asymptotically stable.*

Example 0.3.1 *In order to confirm the results obtained. We consider the following numerical example.*
1/ Assume $x_{-2} = 5.24$, $x_{-1} = 2.54, x_0 = 14, y_{-2} = -7.12, y_{-1} = 0.12, y_0 = 8.16$

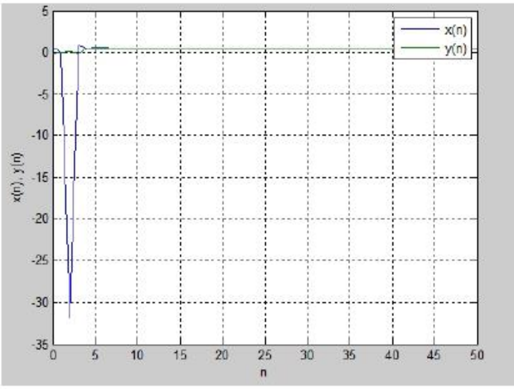

Figure 1- The sequences $(x_n)_{n\geq 0}$ (blue) and $(y_n)_{n\geq 0}$ (green) of solution of the system (12) for $p = 2$ with initial conditions $x_{-2} = 5.24, x_{-1} = 2.54, x_0 = 14, y_{-2} = -7.12, y_{-1} = 0.12$, $y_0 = 8.16$

2/ Assume $x_{-3} = -9.5, x_{-2} = 42.24, x_{-1} = -2.54, x_0 = 14.2, y_{-3} = 17, y_{-2} = 0.12, y_{-1} = 1.2$, $y_0 = 6.16$.

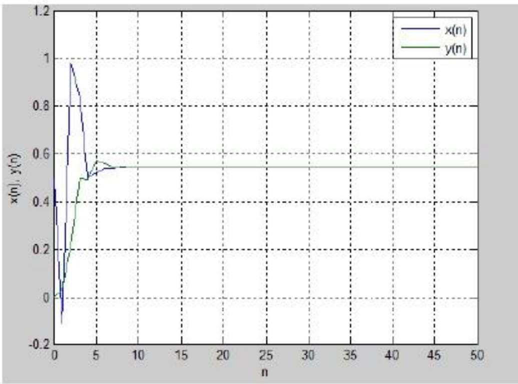

Figure 1- The sequences $(x_n)_{n\geq 0}$ (blue) and $(y_n)_{n\geq 0}$ (green) of solution of the system (12) for $p = 3$ with initial conditions $x_{-3} = -9.5, x_{-2} = 42.24, x_{-1} = -2.54, x_0 = 14.2, y_{-3} = 17$, $y_{-2} = 0.12, y_{-1} = 1.2, y_0 = 6.16$.

From the previous figures ,we show that the sequences $\{x_n, y_n\}_{n\geq 0}$ *of system* (12) *are converges to the equilibrium point* $W := (a, a, \ldots, a, a, a, \ldots, a) \in I^{p+1} \times J^{p+1}$

0.3.2 Second system

In the second part of this work we also study the explicit form and the stability for solutions of the following system of rational difference equations

$$\begin{cases} x_{n+1} = \frac{-1}{y_{n-(p-1)}(x_{n-p}-1)+1}, \\ \\ y_{n+1} = \frac{-1}{x_{n-(p-1)}(y_{n-p}-1)+1} \end{cases} \quad n \in \mathbb{N}_0, \quad p \geq 1 \tag{24}$$

which the initial conditions of the negative index terms :
$x_{-p}, x_{-(p-1)}, x_{-(p-2)}, \ldots, x_0, y_{-p}, y_{-(p-1)}, y_{-(p-2)}, \ldots, y_0 \in \mathbb{R} - F$
with

$$F = \cup \left\{ (x_{-p}, x_{-(p-1)}, x_{-(p-2)}, \ldots, x_0, y_{-p}, y_{-(p-1)}, y_{-(p-2)}, \ldots, y_0) : A_n = 0, B_n = 0, C_n = 0, D_n = 0 \right\}$$

such that

$$\begin{aligned} A_n &= T_{2n-(2p-1)} x_{i-(p+1)} y_{p-i} - (T_{2n-2p} + T_{2n-(2p-1)}) y_{p-i} + T_{2n-(2p-2)} \\ B_n &= T_{2n-(2p-1)} y_{i-(p+1)} x_{p-i} - (T_{2n-2p} + T_{2n-(2p-1)}) x_{p-i} + T_{2n-(2p-2)} \\ C_n &= T_{2n-(2p-2)} y_{p-1} x_{i-(p+1)} - (T_{2n-(2p-1)} + T_{2n-(2p-2)}) x_{i-(p+1)} + T_{2n-(2p-3)} \\ D_n &= T_{2n-(2p-2)} x_{p-i} y_{i-(p+1)} - (T_{2n-(2p-1)} + T_{2n-(2p-2)}) y_{i-(p+1)} + T_{2n-(2p-3)}. \end{aligned}$$

Theoreme 0.3.4 *Let $\{x_n, y_n\}_{n\geq 0}$ be the solutions of system (24). Then for $n = 0, 1, 2, \ldots$; and $p \geq 1$, the forms of $\{x_n, y_n\}_{n\geq 0}$ are given as*
1/ *For $i = 1, 2, \ldots, p$*

$$x_{2pn-i} = \frac{-\left(T_{2n-2p} x_{i-(p+1)} y_{p-i} + (T_{2n-(2p-1)} - T_{2n-(2p-2)}) y_{p-i} + T_{2n-(2p-1)}\right)}{T_{2n-(2p-1)} x_{i-(p+1)} y_{p-i} - (T_{2n-2p} + T_{2n-(2p-1)}) y_{p-i} + T_{2n-(2p-2)}}, \tag{25}$$

2/ *For $i = 1, 2, \ldots, p$*

$$y_{2pn-i} = \frac{-\left(T_{2n-2p} y_{i-(p+1)} x_{p-i} + (T_{2n-(2p-1)} - T_{2n-(2p-2)}) x_{p-i} + T_{2n-(2p-1)}\right)}{T_{2n-(2p-1)} y_{i-(p+1)} x_{p-i} - (T_{2n-2p} + T_{2n-(2p-1)}) x_{p-i} + T_{2n-(2p-2)}}, \tag{26}$$

3/ *For $i = p+1, p+3, \ldots, 2p$*

$$x_{2pn-(i-1)} = \frac{-\left(T_{2n-(2p-1)} y_{p-i} x_{i-(p+1)} + (T_{2n-(2p-2)} - T_{2n-(2p-3)}) x_{i-(p+1)} + T_{2n-(2p-2)}\right)}{T_{2n-(2p-2)} y_{p-i} x_{i-(p+1)} - (T_{2n-(2p-1)} + T_{2n-(2p-2)}) x_{i-(p+1)} + T_{2n-(2p-3)}}, \tag{27}$$

4/ *For $i = p+1, p+3, \ldots, 2p$*

$$y_{2pn-(i-1)} = \frac{-\left(T_{2n-(2p-1)} x_{p-i} y_{i-(p+1)} + (T_{2n-(2p-2)} - T_{2n-(2p-3)}) y_{i-(p+1)} + T_{2n-(2p-2)}\right)}{T_{2n-(2p-2)} x_{p-i} y_{i-(p+1)} - (T_{2n-(2p-1)} + T_{2n-(2p-2)}) y_{i-(p+1)} + T_{2n-(2p-3)}}, \tag{28}$$

where T_n the nth Tribonacci numbers.
Proof
This theorem is proven in the same way as in the proof of theorem (0.3.1)(from a simple induction).

Existence of equilibrium point

We have the following system of equations

$$\begin{cases} \bar{x} = \frac{-1}{\bar{y}(\bar{x}-1)+1} \\ \\ \bar{y} = \frac{-1}{\bar{x}(\bar{y}-1)+1} \end{cases} \tag{29}$$

In (29), and after some computations, we obtained $\bar{x} = \bar{y}$. Therefore, we obtained the following characteristic equation

$$\bar{x}^3 - \bar{x}^2 + \bar{x} + 1 = 0 \tag{30}$$

this equation is having three roots d, e and f which,

$$\begin{cases} d = \frac{1+\sqrt[3]{3\sqrt{33}-17}+\sqrt[3]{3\sqrt{33}+17}}{3} \\ e = \frac{1+\omega\sqrt[3]{3\sqrt{33}-17}-\omega^2\sqrt[3]{3\sqrt{33}+17}}{3} \\ f = \frac{1+\omega\sqrt[3]{3\sqrt{33}-17}-\omega\sqrt[3]{3\sqrt{33}+17}}{3} \end{cases}$$

and $\omega = \frac{-1+i\sqrt{3}}{2}$ is a primitive cube root of unity. Hence the unique real negative equilibrium point of system (24) is given by
$\overline{W} = (d, d, \ldots, d, d, d, \ldots, d) \in I^{p+1} \times J^{p+1}$. Which d is the real root of the characteristic equation : $x^3 - x^2 + x + 1$.

Theoreme 0.3.5 *The equilibrium point of system (24) is locally asymptotically stable.*
Proof
This theorem is proven in the same way as in the proof of theorem (0.3.2), then we obtaid

$$|\lambda_1| = |\lambda_2| = |\lambda_3| = \ldots = |\lambda_{2p}| \simeq 0.40089 < 1.$$

Therfore, the equilibrium point of system (24) is locally asymptotically stable.

Theoreme 0.3.6 *The equilibrium point of system (24)is globally asymptotically stable.*
Proof
This theorem is proven in the same way as in the proof of theorem (0.3.3) then, we obtained

$$\lim_{n\longrightarrow+\infty} x_{2pn-i} = d, \quad \lim_{n\longrightarrow+\infty} x_{2pn-(i-1)} = d$$

and

$$\lim_{n\longrightarrow+\infty} y_{2pn-i} = d, \quad \lim_{n\longrightarrow+\infty} y_{2pn-(i-1)} = d$$

So

$$\lim_{n\to+\infty} (x_n, y_n) = \overline{W} = (d, d, \ldots, d, d, d, \ldots, d) \in I^{p+1} \times J^{p+1}$$

.
Hence, the equilibrium point $\overline{W}$ of system (24) is globally asymptotically stable.

Example 0.3.2 *In order to confirm the results obtained. We consider the following numerical example.*
1/ Assume $x_{-2} = 5.24$, $x_{-1} = 2.54, x_0 = 14, y_{-2} = -7.12, y_{-1} = 0.12, y_0 = 8.16$

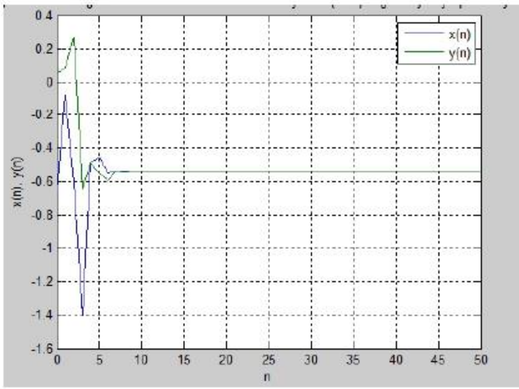

Figure 1- The sequences $(x_n)_{n\geq 0}$ (blue) and $(y_n)_{n\geq 0}$ (green) of solution of the system (12) for $p = 2$ with initial conditions $x_{-2} = 5.24, x_{-1} = 2.54, x_0 = 14, y_{-2} = -7.12, y_{-1} = 0.12,$ $y_0 = 8.16$

2/ Assume $x_{-3} = -9.5, x_{-2} = 42.24, x_{-1} = -2.54, x_0 = 14.2, y_{-3} = 17, y_{-2} = 0.12, y_{-1} = 1.2,$ $y_0 = 6.16$.

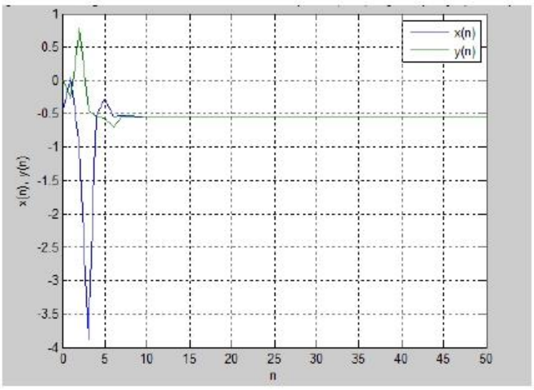

Figure 1- The sequences $(x_n)_{n\geq 0}$ (blue) and $(y_n)_{n\geq 0}$ (green) of solution of the system (12) for $p = 3$ with initial conditions $x_{-3} = -9.5, x_{-2} = 42.24, x_{-1} = -2.54, x_0 = 14.2, y_{-3} = 17,$ $y_{-2} = 0.12, y_{-1} = 1.2, y_0 = 6.16$.

From the previous figures ,we show that the sequences $\{x_n, y_n\}_{n\geq 0}$ *of system* (24) *are converges to the equilibrium point* $W := (d, d, \ldots, d, d, d, \ldots, d) \in I^{p+1} \times J^{p+1}$

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